On Integrability of Type 0A Matrix model in the presence of D brane

Chandrima Paul [1]

Department of Physics,
Indian Institute of Technology Bombay,
Mumbai 400 076, India

Abstract

We consider type 0A matrix model in the presence of spacelike D brane which is localized in matter direction at any arbitrary point. In string theory, the boundary state which in matrix model corresponds to the Laplace transform of the macroscopic loop operator, is known to obey the operator constraints corresponding to open string boundary condition. When we analyze MQM as well as the respective collective field theory and compare it with dual string theory it appears that consistency of the theory requires a condition equivalent to a constraint on the matter part that needed to be imposed in the matrix model. We identified this condition and observed that this has only effect into constraining the macroscopic loop operator so that it projects the Hilbert space generated by the operator to its physical sector at the point of insertion while keeping the bulk matrix model remains unaffected, thereby describing a situation parallel to string theory. We analyzed the theory with uncompactified time and have shown explicitly that the matrix model predictions are in good agreement with the relevant string theory. Next we considered the theory with compactified time, analyzed MQM on a circle in the presence of D brane. We evaluated the partition function along with the constrained macroscopic loop operator in the grand canonical ensemble and showed the free energy corresponds to that of a deformed Fermi surface. We have compared the matrix model features with that of the relevant string theory. We have also shown that the path integral in the presence of D brane can be expressed as the Fredholm determinant. We have studied the fermionic scattering in a semiclassical regime. Finally we considered the compactified theory in the presence of the D brane with tachyonic background. From the collective field theory analysis we have predicted the right structure of the theory in the presence of D brane. We evaluated the free energy in the grand canonical ensemble. We have shown the integrable structure of the respective partition function and it corresponds to the tau function of Toda hierarchy. We have also analyzed the dispersionless limit.

[1] mail:chandrima@phy.iitb.ac.in

Contents

1 Introduction

The two dimensional string theory (see e.g. [1], [2], [3] for reviews) is a very instructive model when we would like to understand the nature of string theory as a complete theory of quantum gravity. This theory has a powerful dual description of $c = 1$ matrix model defined by the simple quantum mechanics of a Hermitian matrix Φ with the inverse harmonic oscillator potential $U(\Phi) = -\Phi^2$ after the double scaling limit.Matrix model is successfully used to describe 2D string theory in the simplest linear dilaton background as well as to incorporate perturbations.

In last decade the $c = 1$ matrix quantum mechanics has received lots of attention because of its new interpretation as the decoupled world volume theory of unstable D0-branes [4–6]. The matrix model dual to type 0 string theories were also proposed in [8,9]. In particular, the matrix model dual of the two dimensional type 0 string gives a non-perturbatively well-defined formulation. For example, the type 0B model is defined by the hermitian matrix model with two Fermi surfaces. The type 0B matrix quantum mechanics (MQM) describes open string tachyons living on the unstable D0-branes, whereas the type 0A MQM describes tachyonic open strings stretched between stable D0- and anti-D0-branes. Upon compactification on Euclidean time, these two matrix models are conjectured to be T-dual to each other. The exact agreement in free energy was found in [9]. Matrix model dual of type 0 string in the flux background was explored in [29,30]. However, unlike $c = 1$ matrix model which can be derived from discretizing the Polyakov action on the string world sheet, such a derivation is not known for type 0 matrix models.

An attempt was made in [13] to obtain the exact form of the macroscopic loop operator in Type 0 string theory. If we consider the bosonic string partition function

$$\int D\phi DX \; exp\left[-\int d^2z[\frac{1}{4\pi}(\partial X\partial X + \partial\phi\partial\phi) + QR\phi + \mu e^{2b\phi}] - \int_{\partial\Sigma} d\xi \; [\frac{Qk\phi}{2\pi} + \mu_B e^{b\phi}]\right], \tag{1.1}$$

the macroscopic loop operator inserts an operator

$$W(t,l) \sim \delta\left(\int_{\partial\Sigma} e^{\phi} - l\right) \cdot \delta(X^0 - t), \tag{1.2}$$

within the path integral [10, 11].

$$\langle W(l)\rangle = Z(l) = \int DXD\phi D[\,\text{ghost}\,]\; \delta\left(\int_{\partial\Sigma} e^{\phi} - l\right)\cdot\delta(X^0 - t)\, f(x,\phi)\, Z(\phi(\sigma), X(\sigma), [\text{ghost}]), \tag{1.3}$$

where f is some wave function for matter ghost and Liouville. The physical meaning of this operator in two dimensional string theory is the presence of a 'Euclidean D-brane' localized in the time direction. To be more precise after we take the Laplace transformation $\int d\phi e^{-\mu_B e^{\phi}}$, we get a D-brane with the Neumann boundary condition in the Liouville direction and the Dirichlet one in the time direction

$$\int \frac{dl}{l} e^{-\mu_B l}\; W_{bos}(t,l) \simeq |B_{(FZZT)}(\mu_B)\rangle_\phi \otimes |D\rangle_{X^0}. \tag{1.4}$$

when we impose the condition that boundary Liouville term is zero.

$$\partial_n \phi + \mu_B e^{b\phi} = 0 \tag{1.5}$$

Where $\partial_n \phi$ denotes the Liouville momentum normal to boundary while along the boundary we have $\partial_t X^o = 0$. Now consider 2D superstring action obtained from extending the bosonic fields to their superspace and expanding the 2D superspace action in terms of the component field,

$$S = \frac{1}{2\pi}\int d^2z[\delta_{\mu\nu}(\partial X^\mu \overline{\partial} X^\nu + \psi^\mu \overline{\partial}\psi^\nu + \overline{\psi^\mu}\partial\overline{\psi^\nu}) + \frac{Q}{4}RX^1] + 2i\mu b^2 \int d^2z(\psi^1\overline{\psi^1} + 2\pi\mu e^\phi) : e^\phi : \tag{1.6}$$

We can also consider the macroscopic loop operator which is the superspace analogue of $W_{bos}(t,l)$, inserts the boundary condition on the fermionic coordinate $\overline{\psi}(\overline{z}) = \eta\psi(z)$ where $\eta = \pm 1$ describes the RR and NS NS sector. Laplace transform of this operator inside the string path integral describes the boundary states NS NS and RR sector. However depending on helicity, in each sector we have two types of boundary states $\epsilon = \pm$ so that we have four types of macroscopic loop operator given by $W^+_{NS}, W^-_{NS}, W^+_R, W^-_R$. The parameter μ_B corresponds to the boundary cosmological constant in the boundary state. Indeed we can show this relation [33,34] by computing one point function on the brane or equally annulus amplitude as shown in [12]. For $c = 1$ matrix model the expression of these operators were obtained and its equivalence to string theory is verified in [10–12]. Author of [13] obtained the expressions of macroscopic loop operator in Type 0B matrix model and also for NS sector of Type 0A matrix model which was verified by calculating the one point function. Now once we understand the duality between noncritical string theory in the linear dilaton background and Matrix model, its natural to ask whether we can understand the string theory with nontrivial background which has an obvious realization in matrix model by adding perturbations which survive in the double scaling limit. There are two ways to change the background of string theory: either to consider strings propagating in a non-trivial target space or to introduce the perturbations . In the first case one arrives at a complicated sigma-model. Not many examples are known when such a model turns out to be solvable. Besides, it is extremely difficult to construct a matrix model realization of a general sigma-model since not much known about matrix operators explicitly perturbing the metric of the target space. Thus, we lose the possibility to use the powerful matrix model machinery to tackle our problems. On the other hand, following the second way, we find that the integrability of the theory in the trivial background is preserved by the perturbations. Also when we study the theory in a nontrivial background in most of the cases the target space metric of such backgrounds is curved and often it incorporates the black hole singularities. In the superstring theories, the supersymmetry allows for some interesting nontrivial solutions which are stable and exact. But the string theory on such backgrounds is usually an extremely complicated sigma-model, very difficult even

to formulate it explicitly, not to mention studying quantitatively its dynamics. The two-dimensional bosonic string theory as well as Type 0 theory are the rare cases of sigma-model where such a dynamics is integrable, at least for some particular backgrounds, including the dilatonic black hole background. A physically transparent way to study the perturbative (one loop) string theory around such a background is provided by the CFT approach. However once we try to understand higher loops or multipoint correlators, we have to address ourselves to the matrix model approach to the 2D string theory .

The 2D string theory has been constructed as the collective field theory [25], [28], in which the only excitation, the massless tachyon, was related to the eigenvalue density of the matrix field. Now consequence of the deformation in eigenvalue density corresponding to deformation in string background at classical limit was studied in [26]. $C = 1$ string theory perturbed by tachyonic mode studied in [32]. Vortex perturbation and its equivalence to sine -Liouville theory was studied in [36], [37]. Its shown that partition function is integrable and have Toda structure. Toda structure and Lax formalism in the context of matrix model described in [31]. Many more works in this direction was done in [15–19, 21–23].

Now its an interesting question to ask that can we study this nontrivial background in the context of dual matrix model in the presence of D brane which are just the Laplace transform of the macroscopic loop operator as we discussed. Open close duality predicts that partition function must have integrable structure. So we consider Type 0A MQM with simplest macroscopic loop operator, which is the operator in NS sector(as prescribed in [13]), which is localized in time direction and with it we show that partition function indeed have an integrable structure.We obtain the string equation.

The plan of this work is in section 2 we are going to consider basic Type 0A MQM in the presence of D brane, with uncompactified time, We are going to introduce a no leakage condition to matrix model which is equivalent to some constraint to boundary state of string theory. We are going to explain its origin as well as its string theoretical interpretation. In section 3 we consider MQM compactified on a circle in the presence of D brane. From path integral approach we are going to show that the partition function can be expressed as Fredholm determinant. We have explicitly evaluated the thermal partition function and have shown that without application of this constraint the partition function diverge. In section 4 we have discussed scattering in semiclassical regime. In section 5 we have considered string theory in the presence of momentum modes and have shown that the partition function in this background have an integrable structure if we apply this constraint.

2 Type 0A MQM in the presence of the D-brane

2.1 Type 0A MQM

Let us start with the MQM of type 0A theory in two dimensions, which is the decoupled world volume theory of (stable) D0 –brane and anti D0–branes. A spacelike D0 –$\overline{D0}$ pair, i.e with Neumann boundary condition in Liouville direction and Dirichlet in matter direction, gives a macroscopic loop observable of the matrix model after Laplace transformation [9]. We are going to consider Type 0A MQM in the presence of this operator and study the relevant physics. Here is a brief review of the Type 0A MQM. In the background with no net D0-brane charges, the matrix model has $U(N) \times U(N)$ gauge symmetry. This is the case we are going to consider. We have the $U(N) \times U(N)$ gauge field A_0 and bifundamental tachyon Φ,

$$A_o = \begin{pmatrix} A & 0 \\ 0 & \tilde{A} \end{pmatrix}, \tag{2.1}$$

$$\Phi = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}. \tag{2.2}$$

The action is

$$\int dtTr\left[(D_oM)^\dagger D_oM + \frac{1}{2\alpha'}M^\dagger M\right], \tag{2.3}$$

where

$$D_oM = \partial_oM + iAM - iM\overline{A}. \tag{2.4}$$

As M is a complex matrix so we denote M by Z, $M^\dagger = \overline{Z}$

$$D_oZ = \partial_oZ + iAZ - iZ\tilde{A} \quad ; \quad (D_oM)^\dagger = \overline{D}_o\overline{Z} = \partial_o\overline{Z} + i\tilde{A}\overline{Z} - i\overline{Z}A. \tag{2.5}$$

Now define

$$Z_\pm = \frac{1}{\sqrt{2\alpha'}}Z \pm D_oZ \quad ; \quad \overline{Z}_\pm = \frac{1}{\sqrt{2\alpha'}}\overline{Z} \pm \overline{D}_o\overline{Z}. \tag{2.6}$$

The Type 0A matrix model action in terms of the light cone variable

$$S = \int dtTr\left[\overline{Z}_+D_AZ_- + Z_+\overline{(D_AZ)} + \frac{1}{2}(\overline{Z}_-Z_+ + \overline{Z}_+Z_-)\right]. \tag{2.7}$$

The gauge field A acts as a lagrange multiplier which projects the theory onto singlet wave functions. Its shown in [9, 15] that type 0A MQM when projected to singlet sector can be represented by non-relativistic free fermions moving in a two dimensional upside-down harmonic oscillator potential

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) - \frac{1}{4\alpha'}(\hat{x}^2 + \hat{y}^2). \tag{2.8}$$

The theory has different independent sectors labeled by net D0-brane charge q, which is the same as the angular momentum $\hat{J} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ [9]. Here we will consider the case

where there is no net D0 –brane charge, namely the $J = 0$ sector. Now with $z, \overline{z} = x \pm iy$; and light cone variable are as defined in (2.6) we have the hamiltonian

$$\begin{aligned} H_o &= -\frac{1}{2}(\hat{z}_+\hat{\overline{z}}_- + \hat{\overline{z}}_+\hat{z}_- - \frac{2i}{\sqrt{2\alpha'}}) \\ &= \mp\frac{i}{\sqrt{2\alpha'}}[z_\pm\frac{\partial}{\partial z_\pm} + \overline{z}_\pm\frac{\partial}{\partial \overline{z}_\pm} + 1]. \end{aligned} \tag{2.9}$$

The commutation relation satisfied by these operators

$$\begin{aligned} [\hat{z}_+, \hat{\overline{z}}_-] &= [\hat{\overline{z}}_+, \hat{z}_-] = 2\frac{i}{\sqrt{2\alpha'}}, \\ [\hat{z}_+, \hat{z}_-] &= [\hat{\overline{z}}_+, \hat{\overline{z}}_-] = 0, \end{aligned} \tag{2.10}$$

so that

$$\hat{\overline{z}}_+ = -\frac{\partial}{\partial z_-} \qquad ; \qquad \hat{z}_- = \frac{\partial}{\partial \overline{z}_+}. \tag{2.11}$$

We have Schrodinger equation

$$i\frac{\partial}{\partial t}\Psi(\overline{z}_+, z_+, t) = \mp\frac{i}{\sqrt{2\alpha'}}[(z_+\frac{\partial}{\partial z_+} + \overline{z}_+\frac{\partial}{\partial \overline{z}_+} + 1]\Psi(z_+, \overline{z}_+, t). \tag{2.12}$$

Note, here we have absorbed the Vandermonde determinant in the wave function so that the wave function ψ in (2.12) describes a fermion.

2.2 Type 0A MQM in the presence of the D brane : The constraint from dual string theory

Consider the type 0A matrix model in the presence of D brane which arises when we insert an operator $e^{\int dt W(t)\delta(t-t_o)}$ in the matrix model path integral where $W(t)$ is the Laplace transform of the macroscopic loop operator([13], [18]). In the dual two dimensional type 0A theory this means that there is one Euclidean $D0$–$\overline{D0}$ brane is localized at time t_0. The branes extends along the Liouville direction after the Laplace transformation. The macroscopic operators can be divided into NSNS and RR sector part such that they correspond to the NSNS and RR sector part of the D-brane boundary state. Moreover,since we know that there are two types of (FZZT-like) boundary states $|B(\epsilon)\rangle$ according to the spin structures there should be two macroscopic operators $W^{(\epsilon)}$ with $\epsilon = \pm$ in each sector. First consider the simplest expression of macroscopic loop operator which is the one in NS NS sector as prescribed in [13] and expressed as $e^{-lM^\dagger M(t_o)}$. Now, consider the Laplace transform of the operator

$$\int \frac{dl}{l} e^{-\mu_B^2 l} e^{-lM^\dagger M} = -Tr\log(1 + \frac{M^\dagger M}{\mu_B^2}) \tag{2.13}$$

$$= -Tr\log(1 + \frac{\overline{Z}Z}{\mu_B^2}) \tag{2.14}$$

$$
\begin{aligned}
&= -\sum \log(1 + \frac{\overline{z}z}{\mu_B^2}) \\
&= -\sum \log(1 + \frac{(z_+ + z_-)(\overline{z}_+ + \overline{z}_-)}{\mu_B^2}) \\
&= -\sum \log(1 + \frac{\overline{z}_+ z_+ + \overline{z}_- z_- + \overline{z}_+ z_- + \overline{z}_- z_+}{\mu_B^2}) \\
&= W(\overline{z}_+, z_+, \overline{z}_-, z_-).
\end{aligned} \tag{2.15}
$$

(Here $\sum$ implies sum over the eigenvalues). Now the macroscopic loop operator for NS^- sector can be expressed as

$$
\begin{aligned}
W &= -\sum \log(1 + \frac{\overline{z}_+ z_+ + \overline{z}_- z_- + \overline{z}_+ z_- + \overline{z}_- z_+}{\mu_B^2}) \\
&= -\sum \{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\frac{\overline{z}_+ z_+ + \overline{z}_- z_- + \overline{z}_+ z_- + \overline{z}_- z_+}{\mu_B^2}]^n \}.
\end{aligned} \tag{2.16}
$$

The path integral over the Euclidean time in the presence of D brane is expressed as

$$
\int \prod dZ_+ dZ_- d\overline{Z}_+ d\overline{Z}_- dA d\tilde{A} e^{-\int dt[\beta L(Z_+,\overline{Z}_+,Z_-,\overline{Z}_-,A,\tilde{A}) - W\delta(t-t_o)]}, \tag{2.17}
$$

apparently implies a shift[2] [3] in the free single fermion hamiltonian $\beta H \to \beta H + W\delta(t-t_o)$ (in Euclidean time). Note here we are going to consider complete quantum theory along with the macroscopic loop operator W. However before proceeding note that the operator $e^{\int dtW\delta(t-t_o)}$ which is localized at $t = t_o$, in general breaks the time translation symmetry of MQM action. So as far as matrix model action in the presence of brane is concerned (as considered in [13]) this is describing leakage of energy exactly at t_o. Now to be more precise consider MQM path integral in the presence of an operator $e^{W(t_o)}$, for which under any infinitesimal variation in time $t \to t + \epsilon(t)$, Ward identity implies [4]

$$
\begin{aligned}
\delta(t-t_o)\delta\langle e^{W(t_o)}\rangle &= -\partial_t \langle H_o(t)\, e^{W(t_o)}\rangle \\
&\Rightarrow \langle\, \delta W(t_o)\, e^{W(t_o)}\,\rangle + \lim_{\epsilon\to 0}\int_{t_o-\epsilon}^{t_o+\epsilon} \partial_t \langle H_o(t)\, e^{W(t_o)}\,\rangle,
\end{aligned} \tag{2.18}
$$

where δW is the variation of W due to infinitesimal time translation at fixed time t_o. The above identity arises when we consider the first order(in ϵ) variation. However $e^{W(t_o)}$ is a coherent source of the operator $W(t_o)$ and on expansion generates infinitely many

[2]Note that as $W(\overline{z}_+, z_+, \overline{z}_-, z_-)$ in any sector expressed in light cone variable, so does not involve any derivative. Hence we can just add it to hamiltonian or lagrangian as a potential localized in t_o

[3]Note, when we are adding the term $W\delta(t-t_o)$ in the expression of the hamiltonian from the operator $e^{W(t_o)}$ it is supposed to add in the hamiltonian the terms like $[\beta \int_{t_o-\epsilon}^{t_o+\epsilon} dtH, \int_{t_o-\epsilon}^{t_o+\epsilon} dt' W\delta(t'-t_o)] + ...$higher commutators $= e^{-W(t_o)}[\beta \int_{t_o-\epsilon}^{t_o+\epsilon} dtH]e^{W(t_o)} + ...O(\beta\epsilon)^2...$ However as around the $\delta-$function, ϵ can be made arbitrarily small i.e $\epsilon << \frac{1}{\beta}$, so upto quasiclassical limit one can just put these terms to zero while in the classical limit these terms are trivially zero.

[4]Here we have used the fact that equation of motion in the presence of an insertion $e^{W(t_o)}$ is satisfied

source W in the path integral. So in principle we sum up the contribution from every order of ϵ this gives rise an operator in the path integral

$$\langle e^{[\epsilon\partial_t H_o(t)+\epsilon\delta W(t)\delta(t-t_o)+\{W(t_o)+\epsilon[\partial_t H_o(t),W(t_o)]+\ldots\text{higher commutators}\}]}\rangle$$

(with proper time ordering of operators). However integrating the argument of exponential over an infinitesimally small interval $t_o-\epsilon$ to $t_o+\epsilon$,time translation invariance implies

$$\int_{t_o-\epsilon}^{t_o+\epsilon} dt\,\partial_t\langle\, H_o(t)\,\rangle + \delta\langle\, W(t_o)\,\rangle = 0. \tag{2.19}$$

(one can verify the commutator terms in this integration will give zero because of time ordering) which is an operator constraint. Note the term $W(t_o)$ which is completely localized at a point t_o essentially creates the effect of boundary in the matrix model action which is defined on infinite real line in the time direction. So any variation of the expectation value of the hamiltonian (from (2.18)) exactly at t_o due to the interaction $[H_o, W\delta(t-t_o)]$ with the source $W(t_o)$ is the signal of leakage of MQM hamiltonian H_o exactly at t_o. So we conclude that the time translation invariance of the path integral implies[5] that in an infinitesimal small interval around t_o we have $\langle\, H_o(t)\,\rangle|_{t_o-\epsilon}^{t_o+\epsilon}+\delta\langle W(t_o)\,\rangle = 0$, where $\delta W(t_o) = [\int_{t_o-\epsilon}^{t_o+\epsilon}[H_o, W(t)\delta(t-t_o)]$ is the variation of $W(t_o)$ due to time translation $t_o \to t_o+\epsilon(t_o)$, which indicates the leakage of energy of the fermionic system. However the inclusion of an operator $W(t_o)$ in MQM action has an interpretation in dual string theory is to create a boundary to string world sheet by insertion of a macroscopic loop localized at $X^o \equiv it_o$ or presence of a boundary state in closed string channel. So the above phenomenon in string theory implies that closed string hamiltonian is undergoing a leakage while being scattered from the boundary state localized at $X = X^o = it_o$! This means that energy from the bulk is flowing out across the boundary or in other words the bulk hamiltonian is not conserved in the presence of boundary! This effect can be visualized in matrix model from the consideration of collective field theory.

Note the path integral with the operator $e^{W(t_o)}$ does have an interpretation that free fermionic state are getting scattered from an operator $e^{W(t_o)}$. The free fermionic states after being scattered becomes permanently changed due to the action of an operator which is the function of the leakage factor $\int dt[\delta W(t)]\delta(t-t_o)$ at $t \geq t_o$. Clearly the scattered state will differ from the incoming state with a term which is function of $\delta W(t_o)$. As the term $\delta W(t_o)$ is not present in the effective hamiltonian [6] or cannot arise in by the time evolution of $\langle H_o\rangle$ w.r.t the complete hamiltonian [7] so it describes a leakage. These deformed states although evolve according to the free fermionic hamiltonian H_o but they can be considered as the superposition of the states which are stationary w.r.t

[5]Note the effect of leakage is observed within an interval $t_o-\epsilon$ to $t_o+\epsilon$. Once we move slightly away from t_o system will evolve according to the conserved hamiltonian H_o

[6]which is given by $W\delta(t-t_o)+e^{-W(t_o)}[\,\beta\int_{t_o-\epsilon}^{t_o+\epsilon}dtH\,]e^{W(t_o)} \sim \beta H_o + W\delta(t-t_o)$ as we mentioned

[7]as $W(t_o) \to W(t_o)+\delta W(t_o)$ is an instantaneous process

a hamiltonian deformed from H_o where the deformation is caused by the leakage as we discussed. Now in collective field theory the fluctuation of collective field from its static value gives a field which corresponds to 2D spacetime tachyon. The action for this fluctuation gives the propagator of a 2D massless scalar. Now in the presence of the operator $e^{W(t_o)}$, the wave function of this scalar field above $t \geq t_o$ although evolve according to the hamiltonian of a 2D massless scalar but will be deformed from the one at $t \leq t_o$ by the action of an operator which is function of the leakage factor $\delta W(t_o)$. As this operator is not present in the path integral so the deformation of the wave function above $t \geq t_o$ must show up as the modification of the propagator from that of a 2D massless scalar ! This can be easily seen from the canonical quantization of 2D massless field and considering the deformation of the wave function.

So we see that although the time translation invariance is maintained by extending Ward's principle to every order but its not giving the right string theory picture! This is because the closed strings which are getting scattered from D brane the scattered states remain the same on shell states w.r.t the hamiltonian same as that for incoming one! So it appears that we must need to impose a constraint in matrix model side in order to extract right string theory from it. Lets briefly go through the string theory scenario and try to understand string theoretical origin of such constraint.

Note as far as the open string world sheet is concerned Dirichlet boundary condition which fixes the matter coordinate $X = X^o \equiv it_o$ at the boundary implies nonconservation of the momentum associated with the matter direction. However the open string action $S_{\rm open}$ and the open string path integral $Z_{\rm open}$ remains invariant under an infinitesimal variation of X which is ensured from the boundary condition

$$\delta X(X^o) = 0 \quad \Rightarrow \quad \delta_X S_{\rm open} = 0 \quad , \quad \delta_X[Z_{\rm open}] = 0, \tag{2.20}$$

where δ_X implies infinitesimal variation in X at every point of the world sheet. Also the conservation of the string hamiltonian is associated with the boundary condition

$$T(z) - \overline{T}(\overline{z}) = 0, \tag{2.21}$$

which ensures there is no leakage of energy at the boundary [8] [39, 40]. Moreover we have the constraint from the current algebra and superpartners of all the above conditions which ensures the conservation of the symmetry generators. Now the string path integral with a Laplace transformed macroscopic loop (which creates the boundary localized at $X = X^o$ with the imposed wave function giving right string one point function)

[8]To explain a bit more, in the presence of the D brane we know the energy of the incoming state is (associated with closed string)is not same as that of the outgoing state in the direction with Dirichlet boundary condition as D brane act as a source. However (2.20) implies we can consider the incoming and the outgoing state as the separate conserved system evolve according to same conserved hamiltonian (but different state) with none suffering any leakage at the boundary

must obey the conditions (2.20,2.21) where the condition (2.20) follows from the function $\delta(X_{\text{boundary}} - X^o)$ present in the macroscopic loop functional. Hence these conditions must show up in the dual matrix model with a D brane. To state more precisely the 2D path integral on a manifold with the macroscopic loop (for the bosonic case which is given by (1.3)) corresponds to a physical state where the respective wave function satisfy WdW equation [10]. WdW equation implies invariance of the wave function under the action of the generator of τ(worldsheet time) translation. However the conservation of these symmetry generators follows from these conditions and hence the respective state must have information about it. So to gain the insight about what these constraints correspond in the matrix model let us look at the closed string channel and express the constraints in terms of the boundary state. Now in the minisuperspace approximation only the zero mode part of the constraints will be relevant and can be expressed as

$$(L_o - \overline{L}_o)|B\rangle = 0 \quad ; \quad \delta X_{\text{boundary}}|B\rangle = 0. \tag{2.22}$$

Note that both the above constraints are followed by their superpartners however as far as zero mode is concerned we have already applied such constraints when we classified macroscopic boundary state according to their spin structure [9]. The second condition essentially describes the zero mode condition

$$(\hat{X} - X^o)|B\rangle = 0 \quad \Rightarrow \quad |B\rangle \equiv \delta(\hat{X} - X^o)|0\rangle. \tag{2.23}$$

In closed siring channel path integral with a Laplace transformed macroscopic loop corresponds to boundary states [39]. This can be seen by expressing the path integral functional $\Psi(X, \phi)$ as a sum over operators(in Minkowskian signature) O_i by state operator mapping where we know that these operators corresponds to Ishibashi states and the wave function $\langle\Psi|O_i\rangle$ gives the one point function. So $|\Psi\rangle$ must be annihilated by the constraints from (2.23) [42]. So the same constraints must be imposed on the state associated with matrix model path integral in the presence of D brane. This is because macroscopic loop operator in matrix model can equivalently be expressed in terms of operators along with the respective wave function where each component is in one to one correspondence with the one in string theory side. So we conclude that the kind of leakage we discussed at the beginning of the section is caused due to absence of any condition equivalent to (2.22) and must be cured once we impose an equivalent condition to matrix model. Lets try to find out the constraint in matrix model. From the first condition in (2.22) along with the one from the zero mode part of the current algebra in Dirichlet boundary state for matter just emphasizes the fact that 2D boundary state or the macroscopic loop operator will correspond to the superposition of primary states/operators expressed in terms of the momentum modes(i.e no winding modes) [39] with a reflection symmetry $P \to -P$ in matter as well as Liouville part [33,39] which is already known in the matrix model [2,28]. The reason we obtain Liouville one point function in exact form from matrix model, is

that in minisuperspace approximation this condition is trivially satisfied and consequently not going to impose any constraint in matrix model side. However the second condition in (2.22), (2.23) or (2.20) is not yet properly understood in the matrix model. More precisely in the presence of macroscopic operator the state from the path integral is represented by an wave function expressed as a functional of bulk d.o.f $\Psi = \Psi(\{X\}, \{\phi\}...)$ and under any infinitesimal transformation $X \to X + \delta X$ we must have

$$\delta\Psi = \int_{\text{boundary}} \delta\hat{X} J(\hat{X})\, \Psi = 0 \tag{2.24}$$

which ensures the bulk conformal invariance and implication of (2.23). Note Ψ is the wave function $\Psi = \Psi(X_{\text{boundary}}, \phi_{\text{boundary}})$ which is an eigenfunction of complete string hamiltonian, representing BRST invariance. Similarly its discussed in [4, 6] that matrix model path integral in the presence of an operator $e^{W(t_o)}$ (which arises by including a probe eigenvalue) is an wave function $\psi = \psi(\overline{z}_\pm z_\pm(t_o)$)which satisfied the Schrodinger equation. However considering the fact that this operator also creates an effect of boundary and ψ is a functional of MQM variables $\psi = \psi(\{\overline{z}_\pm z_\pm(t)\})$ we must have an condition equivalent to (2.24) in matrix model which ensures conservation of MQM hamiltonian in presence of such operator. Naturally no such constraint arises from Liouville d.o.f for the reason as we discussed.

Here first we will find such constraint in the matrix model from somewhat intuitive way, solve it in the context of the matrix model path integral and show its consequence. Finally with the help of it we will come to more rigorous analogy between the string theory and the matrix model scenario in the next subsection.

First note that in the matrix model the matter coordinate X is getting mapped to time(Minkowskian) coordinate, $X \to it$. So we can guess that string theory boundary condition must be reflected in MQM as an overall invariance of the path integral under time translation i.e $\delta X \equiv \delta t$ with no leakage. Note when there is no leakage, under infinitesimal time translation $t \to t + \epsilon(t)$ the variation of path integral is given by $\langle \delta e^{\int dt W(t)\delta(t-t_o)}\rangle$, where δ defines the variation of the operator due to infinitesimal time translation at fixed time $t = t_o$. So string theory boundary condition (2.20) must be reflected in the following constraint in matrix model

$$\delta\langle e^{\int dt W(t)\delta(t-t_o)}\rangle = \langle \delta e^{\int dt W(t)\delta(t-t_o)}\rangle = 0. \tag{2.25}$$

Indeed in the string theory path integral if we expand the macroscopic loop in terms of operators which corresponds to Ishibashi state one can verify that this is the consequence of Ward identity under an infinitesimal transformation $X \to X + \delta X$ which arises on application of the second condition in (2.20) and we have already mentioned it in an alternative way in (2.24).

In next subsection we will show that the consequence of this condition are in exact agreement with that of string theory. To understand the impact of this condition in MQM

first we need to write down the Schrodinger equation and study the Hilbert space. We have the time dependent Schrodinger equation for a single fermion [9] in Minkowskian time

$$\begin{aligned}
&[i\frac{\partial}{\partial t} - i\delta(t-t_0)W(\overline{z}_+, z_+, \overline{z}_-, z_-)]\Psi(\overline{z}_\pm z_\pm, t) \\
&\quad = \mp\frac{i}{\sqrt{2\alpha'}}[(z_+\frac{\partial}{\partial z_\pm} + \overline{z}_+\frac{\partial}{\partial \overline{z}_\pm} + 1]\Psi(z_\pm, \overline{z}_\pm, t).
\end{aligned} \tag{2.26}$$

For t away from t_o we have time independent Schrodinger equation

$$i\frac{\partial}{\partial t}\psi(\overline{z}_\pm z_\pm, t) = \mp\frac{i}{\sqrt{2\alpha'}}[(z_\pm\frac{\partial}{\partial z_\pm} + \overline{z}_\pm\frac{\partial}{\partial \overline{z}_\pm} + 1]\Psi(\overline{z}_\pm z_\pm, t) = E\psi(\overline{z}_\pm z_\pm, t). \tag{2.27}$$

The free fermion solution with energy E is

$$\psi^E_{o\pm}(z_\pm, t) = e^{-iEt}e^{\mp i\frac{\phi_o(E)}{2}}(\overline{z}_\pm z_\pm)^{\pm iE - \frac{1}{2}}, \tag{2.28}$$

where we have chosen $\alpha' = 2$ and $\phi_o(E)$ is determined from biorthogonal property (discussed in the Appendix) and given by

$$e^{i\phi_0(E)} = \frac{\Gamma(iE+\frac{1}{2})}{\Gamma(-iE+\frac{1}{2})}. \tag{2.29}$$

Now consider the commutation relation

$$\begin{aligned}
[H_o, \hat{z}_+] &= -i \quad ; \quad [H_o, \hat{\overline{z}}_+] = -i, \\
[H_o, \hat{z}_-] &= \quad i \quad ; \quad [H_o, \hat{\overline{z}}_-] = i.
\end{aligned} \tag{2.30}$$

This implies $\hat{\overline{z}}_+\hat{z}_+$ and $\hat{\overline{z}}_-\hat{z}_-$ when acts on a state $|E\rangle$ expressed as $\hat{\overline{z}}_+\hat{z}_+|E\rangle = |E-i\rangle$ and $\hat{\overline{z}}_-\hat{z}_-|E\rangle = |E+i\rangle$. These states can be represented as $|E \pm ni\rangle$. These describe a different Hilbert space [20] which can be understood as their inner product with the state $|E\rangle$ either diverge or zero. These states actually can be identified with the discrete tachyonic states over the matrix model ground state [20, 28].

Now to the meaning of the constraint. First we will find the expression for the constraint and then solve it in the context of the matrix model path integral for the macroscopic loop operator we considered or in the more complicated case to reach to the right expression free energy. Consider the v.e.v of the operator in single fermionic state which is given by

$$\langle e^{W(t_o)}\rangle = \langle e^{\log(1+\frac{\overline{z}_+z_+ + \overline{z}_-z_- - 2H_o}{\mu_B^2})}\rangle = \langle(1+\frac{\overline{z}_+z_+ + \overline{z}_-z_- - 2H_o}{\mu_B^2})\rangle. \tag{2.31}$$

[9] When we consider the insertion of β factor, for the macroscopic loop operator we have the expression $\frac{1}{\beta}\hat{W}$, however when we rewrite the Schrodinger equation in terms of the eigenvalues x,y which corresponds to the real and imaginary part of eigenvalue z we have the Schrodinger equation

$$[\frac{1}{\beta^2}\frac{\partial^2}{\partial x^2} + x^2 + \frac{1}{\beta^2}\frac{\partial^2}{\partial y^2} + y^2 + \frac{1}{\beta}\hat{W}\delta(t-t_o)]\psi = E\psi$$

In the double scaling limit we take $x, y \to \sqrt{\beta}x, \sqrt{\beta}y$ [1], which gives the Schrodinger equation (2.12)

(Here we have shown the expectation value w.r.t the single fermionic state. This we could do because the theory is projected singlet sector and N fermionic state is just the direct product of each). So (2.25) along with (2.30) gives the constraint

$$\langle \overline{z}_+ z_+(t_o) - \overline{z}_- z_-(t_o) \rangle = 0. \tag{2.32}$$

As the constraint is exclusively on the variable associated with W so it must has effect in constraing the Hilbert space created by W at t_o. Note for the macroscopic loop operator in any other sector, in general the variation of $\langle e^{W(t_o)} \rangle$ can be expressed as

$$\langle e^{W(t_o)} \delta W(t_o) \rangle = 0 \qquad \Rightarrow \qquad \langle e^{W(t_o)} [\overline{z}_+ z_+(t_o) - \overline{z}_- z_-(t_o)] \rangle = 0. \tag{2.33}$$

As the path integral with the uncompactified time essentially describes the transition amplitude from the initial state $|\overline{z}_\pm z_\pm(t_i)\rangle$ to the final state $|\overline{z}_\pm z_\pm(t_f)\rangle$, so the condition (2.25) remain true for any arbitrary variation in time implies that we have

$$\langle \psi_f | \delta e^{W(t_o)} | \psi_i \rangle = 0, \tag{2.34}$$

between any two physical states $|\psi_i\rangle$, $|\psi_f\rangle$ which according to the Hilbert space described by (2.30) will be of either form $|\overline{z}_+ z_+, t\rangle$ or $|\overline{z}_- z_-, t\rangle$. Now recall the expression of the inserted macroscopic loop operator $W(t_o)$ in (2.15) which is expressed in the form (2.31). While the $\hat{H}_o$ in the expression of W keeps the underlying state invariant the $\hat{\overline{z}}_+ \hat{z}_+$ and $\hat{\overline{z}}_- \hat{z}_-$ act on the vacuum to create the states $|E = -i\rangle$, $|E = i\rangle$ or in other words the macroscopic loop operator (2.15) due to its $\overline{z}z$ field acts on the vacuum to create a left moving as well as right moving state. So the constraint essentially relates the two exactly at t_o by putting the following constraint inside the path integral,

$$\overline{z}_+(t_o) - \overline{z}_-(t_o) = 0 \quad ; \quad z_+(t_o) - z_-(t_o) = 0, \tag{2.35}$$

The physical meaning of the above condition is that the quantum fluctuations of the variables $z_+(\overline{z}_+)$ and $z_-(\overline{z}_-)$ around its classical value appears to be identical at t_o. This in turn implies that the wave functions of the states which are being created by the action of $\hat{\overline{z}}_+ \hat{z}_+$ and $\hat{\overline{z}}_- \hat{z}_-$ over the ground state in $\overline{z}_+, z_+$ and $\overline{z}_-.z_-$ representations respectively , appeared to be indistinguishable exactly at t_o. Note that in the above constraint (2.35), the l.h.s will not remain invariant once we move away from t_o. In otherwords if we express them in terms of the respective operators(which acts on vacuum), we see that l.h.s does not commute with the free hamiltonian H_o (2.9). So once we move away from the point $t = t_o$ we have our physical Hilbert space described by the free hamiltonian H_o and the relation (2.30) so that again the left and right moving states created by $\hat{W}$ will appear to be distinguishable. In next few steps we will see that the condition (2.35)when acts inside the path integral it just has a meaning to project the operator W and the Hilbert

space generated by it to its physical sector while keeping the bulk physics unaffected! So with the constraint we can express the path integral somewhat schematically as [10] [11]

$$\int d\overline{z}_+ dz_+ d\overline{z}_- dz_- e^{-\int_{-\infty}^{\infty} dt[\beta L - W(\overline{z}_+, z_+, \overline{z}_-, z_-; t_o)]} = \int \prod_{t\leq t_o} d\overline{z}_+ dz_+ d\overline{z}_- dz_- e^{-\int_{t_o}^{\infty} dtL}$$
$$\int d\overline{z}_+(t_o) dz_+(t_o) d\overline{z}_-(t_o) dz_-(t_o) \quad \delta(\overline{z}_+(t_o) - \overline{z}_-(t_o))\delta(z_+(t_o) - z_-(t_o)) e^{W(\overline{z}_+ z_+, \overline{z}_- z_-, H_o; t_o)}$$
$$\int \prod_{t\geq t_o} d\overline{z}_+ dz_+ d\overline{z}_- dz_- e^{-\int_{-\infty}^{t_o} dtL} = \int d\overline{z}_+ dz_+ d\overline{z}_- dz_- e^{-\int_{-\infty}^{\infty} dt[\beta L - W(\overline{z}_\pm z_\pm(t_o), H_o)]}. \tag{2.36}$$

Its important to note that the effect of the constraint is only to project W at its physical sector without imposing any boundary condition to original lagrangian which happens due to $\delta(t - t_o)$ factor as we explained. Projection implies we can get same path integral expression or the transition amplitude by expressing W either in $\overline{z}_+ z_+$ or in $\overline{z}_- z_-$ mode which happens due to the fact that wave function associated with the either mode appears to be same at t_o. Its also important to note that when we consider the original expression of W (2.15), the quantum fluctuations of the varibles $z_+(z_-)$ and $\overline{z}_+(\overline{z}_-)$ are constrained by (2.35) and in that sense when we consider the complete Hilbert space as described by (2.30). $\hat{z}_+(\hat{z}_-)$ and $\hat{\overline{z}}_+(\hat{\overline{z}}_-)$ are not ordinary operators. However the implication of (2.36) is that we can replace the theory with the expression of W (as in (2.15)) by the projected one $W \to W_{\rm proj} = W(\hat{\overline{z}}_\pm \hat{z}_\pm, H_o)$ and in the theory with $W_{\rm proj}$ these operators act as ordinary operators and one can evaluate the partition function in a formal way in MQM by using $W_{\rm proj}$ instead of W to give the right transition amplitude as in the original theory with the constraint(2.35).

Now we will show that how the constraint is solved in the context of the path integral leads to the right expression of the partition function. Note in (2.34) $|\psi_i, t_i\rangle$ can be expressed in the free fermionic Hilbert space $|E\rangle$ while the constraint (2.25) implies the macroscopic loop operator which has expression $W(\overline{z}_+ z_+(t_o), H_o)$ or $W(\overline{z}_- z_-(t_o), H_o)$ leads to the most general expression of the final state $|\psi_f\rangle$ is $\sum c_{o;E}|E\rangle + \sum c_{n;E}|E + ni\rangle$ or $\sum c_{o;E}|E\rangle + \sum c_{-n;E}|E - ni\rangle$. As the tachyonic states have imaginary energy so the out state with indefinite number of tachyons will contribute to the path integral. So (2.34)

[10] above identity can arise by inserting $\delta(\overline{z}_+(t_o) - \overline{z}_-(t_o))\delta(z_+(t_o) - z_-(t_o))$ in the path integral, operating $\overline{z}_- z_- = \frac{\partial}{\partial z_+}\frac{\partial}{\partial \overline{z}_+}$ on the inner product $\langle \overline{z}_+ z_+(t_o + dt_o)|\overline{z}_+ z_+(t_o)\rangle$. Its important to note that due to $\delta(t - t_o)$, inside the path integral we can make the time interval around t_o arbitrarily small i.e $dt << \frac{1}{\beta}$ so that around t_o, $e^{-\beta\int_{t_o-\epsilon}^{t_o+\epsilon} dtL}$ will just be identity and hence the effect the constraint is only to modify W without affecting the MQM lagrangian L and hence no boundary condition will be imposed on the variables in original lagrangian at $t = t_o$. Finally integrating over $z_-(t_o), \overline{z}_-(t_o)$ where the $\delta -$ function gives $\overline{z}_-, z_- = \overline{z}_+, z_+$, leads to the above expression of W. As W becomes independent of $\overline{z}_-, z_-$ so we can just express the integral as a continuous integral. Also note in the expression of $W(\hat{\overline{z}}_+ \hat{z}_+, H_o)$,the action of H_o on any intermediate state in the path integral gives the constant and it can be alternatively given by $\frac{\partial}{\partial t}$. So it will not be affected by the constraint.

[11] note that here path integral is expressed in a schematic way omitting the angular factors. One can verify that even the inclusion of angular factor will not change the picture

now can be expressed as a trivial identity

$$\sum_{n=0}^{\infty} c_{-n;E}\langle E-ni; f|\exp[W(\overline{z}_+z_+, H_o, (t_o))]|E\rangle = \sum_{n=0}^{\infty} c_{n;E}\langle E+ni; f|\exp[W(\overline{z}_-z_-, H_o, (t_o))]|E\rangle. \tag{2.37}$$

As $\psi(\overline{z}_+z_+, E)$ and $\psi(\overline{z}_-z_-, E)$ are the same wave function in different representation so we see in the above l.h.s and r.h.s are the same expression, expressed in different representation. Now before coming to the string theoretical interpretation of all the matrix model events first we will derive the expression of the wave function(physical) for $t \geq t_o$. Integrating (2.26) over the infinitesimally small interval around $t = t_o$ and following the constraint from (2.35,2.36) we have

$$i\Psi(z_\pm, \overline{z}_\pm, t)|_{t_0+\epsilon} - i\Psi(z_\pm, \overline{z}_\pm, t)|_{t_0-\epsilon} = iW_{\text{proj}}\Psi(z_\pm, \overline{z}_\pm, t_0) \tag{2.38}$$

$$. = -\sum i\log(1 + \frac{2\hat{\overline{z}}_\pm\hat{z}_\pm + \hat{\overline{z}}_+\hat{z}_- + \hat{\overline{z}}_-\hat{z}_+}{\mu_B^2})\Psi(z_\pm, \overline{z}_\pm, t_o) \tag{2.39}$$

So exactly at $t = t_o$ the wave function is given by

$$\lim_{\epsilon\to 0}\Psi_>(z_\pm, \overline{z}_\pm, t_o) = (1 - W(\hat{\overline{z}}_\pm\hat{z}_\pm, H_o))\Psi_o(z_\pm, \overline{z}_\pm, t_o) \tag{2.40}$$

So for the macroscopic loop operator in the NS NS sector we have the physical wave function for $t \geq t_o$

$$\Psi_>(\overline{z}_\pm z_\pm, t_o) = [1 - \log(1 - \frac{2\hat{\overline{z}}_\pm\hat{z}_\pm(t_o) - 2H_o}{\mu_B^2})]\Psi_o(\overline{z}_\pm z_\pm, t) \tag{2.41}$$

Now we need to find the wave function at $t > t_o$. In order to do so first note that for $t \geq t_o$ wave function evolves according to the free hamiltonian H_o. So at the first sight it appears that the wave function at $t \geq t_o$ is given by the one obtained from the time evolution from $\Psi_>(\overline{z}_\pm z_\pm, t_o)$. It is given by

$$\Psi_>(\overline{z}_\pm z_\pm, t) = [1 - \log(1 - \frac{2\hat{\overline{z}}_\pm\hat{z}_\pm(t) - 2H_o}{\mu_B^2})]\Psi_o(\overline{z}_\pm z_\pm, t) \tag{2.42}$$

Now although exactly at $t = t_o$, $\hat{W}$ is expressed in the form $W(\hat{\overline{z}}_\pm\hat{z}_\pm, H_o)$ but once we move above t_o, W can in principle be expressed in terms of both $\hat{\overline{z}}_+\hat{z}_+$ and $\hat{\overline{z}}_-\hat{z}_-$ as both of them are related by the constraint (2.35) at t_o. More precisely these states are given by

$$|\psi_f\rangle = \sum c_{mn}(E)(\hat{\overline{z}}_+\hat{z}_+)^m(\hat{\overline{z}}_-\hat{z}_-)^n|E\rangle, \tag{2.43}$$

So following the previous discussion $|\psi_f\rangle$ is expected to be given by

$$(1 - W(\hat{\overline{z}}_+\hat{z}_+(t), \hat{\overline{z}}_+\hat{z}_+(t)H_o)), \Psi_o(z_\pm, \overline{z}_\pm, t). \tag{2.44}$$

We can express these states as the one time evoluted from the wave function from $t = t_o$. Although it appears that exactly at $t = t_o$, $|\psi_f\rangle$ and $|\Psi_>\rangle$ are of similar structure but they

are not same as time evolution property of $\hat{\tilde{z}}_+$ and $\hat{z}_+$ are different and once we move away from t_o we have our original Hilbert space. Finally note that at $t = t_o$ fermionic states are being converted from ψ_o to $\psi_>$ so the change of fermion number N must be measured from the variation of the transition amplitude at $t = t_o$ i.e in terms of the macroscopic loop operator described in (2.15) we have

$$\frac{dN}{dt} = \frac{d}{dt}\langle\psi_>(t_f)|\psi_o(t_i)\rangle = \langle\frac{dW}{dt}\rangle|_{t=t_o}, \tag{2.45}$$

Where $t_i \to -\infty$ and $t_f \to \infty$. The average $\langle\frac{dW}{dt}\rangle|_{t=t_o} = 0$ described above is w.r.t the path integral. So in order to have the fermion number unchanged we must need to impose the condition

$$\langle\frac{dW}{dt}\rangle|_{t=t_o} = 0. \tag{2.46}$$

Note that this is just a condition parallel to (2.25) and have an effect to project W as described above. In the next part of this section we are going to see that all these matrix model events are exactly in one to one correspondence to the string theory.

2.3 String theoretical interpretation

In this section we will see that the constraint (2.25,2.35) we imposed leads to right string theoretical result. First The boundary state of 2D spacelike brane is expressed as $|B_{(\text{SuperFZZT})}(\mu_B)\rangle_\phi \otimes |D\rangle_{X^0}$ where $|D\rangle_{X^0}$ for NS NS sector [39] is expressed as $\mathcal{N}\int_{-\infty}^{\infty} dPe^{iPX^o}|P\rangle$ ($\mathcal{N}$ is the normalization factor) which in terms of the vertex operator can be written as

$$|B\rangle_X = \mathcal{N}\int_{-\infty}^{\infty} dP_m[e^{iP_m(X-X^o)}]|0\rangle + \text{descendants}. \tag{2.47}$$

The string endpoints are localized at $X = X^o$ and the matter part of the boundary state form the representation of $\delta(\hat{X} - X^o)|0\rangle$, satisfying (2.22). Boundary state of Super Liouville NS NS sector is given by

$$|B\rangle_L = \mathcal{N}_l\int_0^{\infty} dPU(P_l)|P_l\rangle \quad ; \quad U(P_l) = \frac{\pi\cos 2\pi s P_l}{\sinh(\pi P_l)} \quad ; \quad |P_l\rangle = (1 + \frac{L_{-1}\tilde{L}_{-1}}{P_l^2 + M^2} + ...)|v_{P_l}\rangle \tag{2.48}$$

where $|v_{P_l}\rangle$ primary macroscopic state associated with a vertex $e^{(\frac{Q}{2}+iP_l)\phi}$, M is the mass of intermediate propagating mode.

The boundary state is the direct product of the matter and Liouville part along with the ghost factors. Setting $P_m = P_l = P$ we have the primary part of the boundary state without ghost excitation mode, which is the superposition of the tachyonic field can be expressed in terms of the operators from the state operator mapping as

$$\int dPU(P)[e^{iP(X-X^o)} + e^{-iP(X-X^o)}]e^{(\frac{Q}{2}+iP)\phi}. \tag{2.49}$$

Now in the 2nd quantized matrix model we can express the macroscopic loop operator as

$$W(l,t) = \int e^{l\bar{z}z}\psi^\dagger\psi = \int d\tau e^{l\cosh^2\tau}\partial_\tau\eta(l,t), \tag{2.50}$$

where τ is the time of flight coordinate obtained from reparameterization $\overline{z}z = r^2 \quad ; \quad r^2 \sim \cos^2 h\tau$, $\psi^\dagger\psi \sim \partial\tau\eta(l,t)$ and η is the massless bosonic field $(\partial_t^2 - \partial_\tau^2)\eta(t,\tau) = 0$ which corresponds to the tachyon in the string theory at asymptotic τ [2,25,28]. η corresponds to the fluctuation of the collective field $\phi = \phi_o + \partial_\tau\eta$. η satisfies Dirichlet boundary condition in τ direction [11], [28]. However note that when associated with the Laplace transformed macroscopic loop operator operator W which correspond to D brane boundary state, η is no longer an ordinary state but it corresponds to an Ishibashi state. Here we show that the constraint we imposed (2.25,2.35) leads to the right matter one point function from η what is expected from string theory. Consider

$$\eta(\tau,t) = \int_{-\infty}^{\infty} \frac{dp}{p}\tilde{\eta}(p)(a(p)e^{-ipt} + b(p)e^{ipt})\text{sin}(p\tau). \tag{2.51}$$

In order to find the exact t dependence consider the fact that the implication of (2.46) in the context of collective field theory is

$$\delta_t\phi|_{t=t_o} = 0, \tag{2.52}$$

where as before δ_t implies variation of the collective field ϕ due to infinitesimal variation in t, at fixed t.

So (2.52) implies in (2.51) we have

$$a = e^{ipt_o} \quad ; \quad b = e^{-ipt_o} \tag{2.53}$$

So (2.51) is expressed as $\eta(\tau,t) = \int_{-\infty}^{\infty} \frac{dp}{p}\tilde{\eta}(p)(e^{-ip(t-t_o)} + e^{ip(t-t_o)})\text{sin}p\tau$. Now integrating over τ we have $W(p,t) = \frac{e^{-l\mu}}{2}pK_{ip}(\mu l)\tilde{\eta}(p)(e^{-ip(t-t_o)}+e^{ip(t-t_o)})$ where $K_{ip}(\mu l)$ is the Bessel's function which is the macroscopic wave function satisfying WdW equation [13]. This on Laplace transform $\int \frac{dl}{l}e^{-\mu_B^2 l}$ can be expressed as

$$W(p,t) = U(p)\tilde{\eta}(p,t) \quad ; \quad U(p) = \frac{\pi\text{cos}2\pi sp}{\text{sin}h(\pi p)} \tag{2.54}$$

where we have

$$\mu_B^2 = 2\sinh^2(\pi s)|\mu| \quad (\epsilon\cdot\text{sign}(\mu) < 0), \qquad \mu_B^2 = 2\cosh^2(\pi s)|\mu| \quad (\epsilon\cdot\text{sign}(\mu) > 0). \tag{2.55}$$

Note (2.25) and (2.52)are just the same constraint expressed in the 1rst and 2nd quantized formalism. Given the form of η, on Laplace transform of (2.50) and on inverse Fourier transform of (2.54) we can express W as

$$W_p(\tau,t) = U(p)\tilde{\eta}(p)[e^{ip(t-t_o)}+e^{-ip(t-t_o)}] = U(p)[\partial_\tau\eta_p(\tau+(t-t_o))+\partial_\tau\eta_p(\tau-(t-t_o))] \tag{2.56}$$

where the suffix p implies pth component. We know η corresponds to the space time tachyonic field in the asymptotic τ region which in the string theory side describes asymptotics

of Liouville field ϕ which describes vanishing Liouville wall. So we see that each component with momentum p in the expansion of W describes the tachyonic operator $\tilde{\eta}(p)$ dressed with Liouville and matter wave function (2.47,2.48) where W is symmetric under $P \to -P$. So these are just in one to one correspondence with the state obtained in the expansion of the boundary state in (2.49). So (2.56) is just the same as (2.49) when we identify the physical d.o.f. More interestingly if we try to extrapolate η in the region $P_m \neq P_L$ with the same τ dependence as in (2.51), from (2.52,2.51,2.56) we have

$$\begin{aligned} \int \frac{dl}{l} e^{-\mu_B l} W(l) &= [\int_0^\infty dP_l U(P_l) \eta_l(P_l)] \times [\int_{-\infty}^\infty dP_m e^{iP_m(t-t_o)}] \\ &= [\int_0^\infty dP_l U(P_l) \eta_l(P_l)] \times \delta(t-t_o), \end{aligned} \tag{2.57}$$

where η_l implies Liouville part of η. This has a very similar structure with that of 2D boundary state

$$|B\rangle = |B_{\text{Liouville}}\rangle \otimes |B_{\text{matter}}\rangle = \int_0^\infty dP_l U(P_l)|P\rangle \otimes \delta(\hat{X} - X^o)|0\rangle. \tag{2.58}$$

However this is a mere extrapolation without any physical justification. Finally we see the effect of the constraint (2.25) is only to give the correct form of the matter wave function while keeping the Liouville wave function unchanged as we discussed in section 2.2.

Finally lets explain the meaning of the projection $W \to W_{\text{proj}}$ as described in (2.36),(2.37). Note both Liouville as well as the matter part of the boundary state is symmetric under interchange of $P \to -P$ (2.48). So projecting the theory to either right moving or left moving part i.e in the boundary state keeping either the left moving or the right moving part while expressing the superposition of left and right moving state as only left or only right moving state (i.e flipping the sign of p where necessary)in the incoming or the outgoing sector yield the same transition amplitude as the original one. Overall time translation invariance of the projected theory follows from the consideration of either left or the right moving part as the image of the other.

Lets come to a precise analogy between MQM, 2nd quantized matrix model or collective field theory and string theory, which arises as a consequence of the constraint. We know the macroscopic loop operator can be expanded in terms of microscopic operators. To see that in dual matrix model consider $W_{\text{proj}}(\hat{\bar{z}}_\pm \hat{z}_\pm, H_o)$,acting on ground state

$$\log(1 + \frac{\hat{\bar{z}}_+ \hat{z}_+ + \hat{\bar{z}}_- \hat{z}_- - 2H_o}{\mu_B^2})|\mu\rangle = \sum a_{mn}(\mu)[\frac{\hat{\bar{z}}_+ \hat{z}_+}{\mu_B^2}]^n [\frac{\hat{\bar{z}}_- \hat{z}_-}{\mu_B^2}]^m |\mu\rangle \tag{2.59}$$

Essentially the above which is described in Minkowskian time corresponds to a coherent state. Also the above is an expansion in terms of microscopic operator. This is evident from the correspondence between the operator $(\hat{\bar{z}}_\pm \hat{z}_\pm)^n$ and the vertex operator for discrete tachyonic states. [28]. From the previous discussion it follows that when we consider the space time wave function for each component in W, we found an effect of boundary

exactly at $X = X^o(\mathrm{or}, t = t_o)$. More precisely at this point the state associated with each component of boundary state in (2.49) or the states appear on expansion of W (2.56) in collective field theory, expressed with an wave function which has only Liouville part. So the left and right moving state appears to be identical at $X = X^o(\mathrm{or}\, t = t_o)$. Now in order to understand this effect in MQM note that the macroscopic loop operator $W(\hat{\overline{z}}_+\hat{z}_+, \hat{\overline{z}}_-\hat{z}_-, H_o)$ acting on ground state, essentially create a left moving as well as right moving state (2.59). The constraint (2.35) inside the path integral have an interpretation in MQM that at $t = t_o$ we have

$$[\hat{\overline{z}}_+\hat{z}_+]^n||\mu\rangle \equiv [\hat{\overline{z}}_-\hat{z}_-]^n|\mu\rangle. \tag{2.60}$$

i.e the left and right moving states which are created by the action of $\hat{W}$ on MQM ground state appear to be exactly identical at t_o (but different when we move away from t_o). This leads to the projection $W \to W_{\mathrm{proj}} = W(\hat{\overline{z}}_\pm\hat{z}_\pm(t_o), H_o)$ in the path integral (2.36). More explicitly the created states obey $\psi_+(\overline{z}_+z_+(t_o); E = ni) = \psi_-(\overline{z}_-z_-(t_o); E = -ni)$. This is supported from (2.35) and give the justification for (2.37). Again lets emphasize on the fact that these constraints no way affects the free fermionic states as well as the discrete tachyonic states of the original Type 0A matrix model. This distinction just corresponds to that of the closed string states associated with the boundary states and the free closed string states in the bulk.

Now once move away from t_o we are back to our original scenario described by (2.30) with the distinguishable states described by (2.41,2.44). An important point to note here is that when a bulk operator approaches to the boundary we encounter a singularity [40]. Similarly in MQM we have the ordering ambiguity of the operators arises when we move from the region $t = t_o$ to $t \neq t_o$ on time evolution of (2.40) to (2.44). If we consider the entire space of N fermions $N \to \infty$ this leads to singularity. Now in order to find the exact coefficient of superposition of the states from (2.41) and (2.43) above t_o we must remember the fact that these states just show up as indistinguishable at t_o while away from t_o they are distinct. So the coherent state above t_o must be given by (2.44). From the discussion of [2, 11, 28] these operators corresponds to special tachyonic states and higher Virasoro primaries.

Now in this context we need to identify the open string operator or the boundary operator. Note that exactly at $t = t_o$ for the states/operators from the string theory(2.47), collective field theory (2.56) or from the MQM (2.60) (when we set the origin of time at t_o), they do not show any explicit time dependence exactly at t_o. These operators can either be viewed as the one obtained from the one at $t \neq t_o$ by time evolution as we discussed throughout or the one without any explicit time dependent part. The time independent one must correspond to the states/operators which are extended in Liouville direction but localized in matter direction. Hence they should be identified with the open string operators.

Finally let us briefly tell about matrix model string theory correspondence of the complete scenario we just obtained. Theory describes free fermionic state as well as coherent state where the coherent state is strongly localized at $t = t_o$. The closed string state turned into a coherent state at t_o as described by (2.39), which follows the view of [5, 44]. The transition amplitude can be read from collective field theory.

As the collective field correspond to a spacetime tachyon so from (2.56) we obtain the closed string emission/absorption amplitude, which is giving the one point function in mini superspace approximation

$$\langle \hat{W}\,\hat{V}(P)\rangle = U(P)e^{-iPt_o.} \tag{2.61}$$

where V(P) is the operator representing tachytonic vertex and the above relation can be viewed on evaluation of the above in 2nd quantized field theory [2]. Now the state in (2.41) while show up as time dependent state w.r.t the free hamiltonian H_o, they are stable w.r,t an effective hamiltonian $H_{\text{eff}} = e^{-W}H_o e^{W}$. The effect of the macroscopic loop operator is to cause transition of a fermion from Fermi level to above. At the double scaling limit $\beta \to \infty$ the transition amplitude for a single fermion is given from (2.28,2.40) by

$$\langle \psi_>(E', t_f)|\psi(E, t_i)\rangle \sim \langle \psi_o(E')|W|\psi_o(E)\rangle, \tag{2.62}$$

with $t_i(t_f) = -\infty(\infty)$. The presence of D brane will change the Fermi level $\mu \to \mu'$. So from the 2nd quantized theory we can see that the transition amplitude $|\mu\rangle \to |\mu'\rangle$ is in accordance with [4], [38].

$$\langle \mu'|W(l, t)|\mu\rangle = e^{-i\delta(\mu-\mu')}e^{i(\mu-\mu')t}K_{i(\mu-\mu')}(\sqrt{\mu}l). \tag{2.63}$$

Note the transition amplitude is time independent which is evident from matter one point function (2.56) and it signifies the fact that we have stable D brane.

3 Type 0A MQM on a circle in the presence of D brane

3.1 Evaluation of the free energy of Type 0A matrix model on a circle

We consider Type 0A matrix model compactified on a circle of radius R. As considered in the previous section, there is no net background D0 brane charge and hence it is described with $U(N) \times U(N)$ gauge symmetry. The partition function in terms of the light cone variables is given by

$$\int dZ_+dZ_-d\overline{Z}_+d\overline{Z}_-dAd\tilde{A}e^{-\beta\int_0^{2\pi R}dtTr\left[\overline{Z}_+D_AZ_-+Z_+\overline{(D_AZ_-)}+\frac{1}{2}(\overline{Z}_-Z_++\overline{Z}_+Z_-)\right]}, \tag{3.1}$$

where $D_A Z = \partial_t Z + i[AZ - Z\tilde{A}]$ and A is as given in (2.1). Now we fix the gauge $\partial_t A = 0$, which sets A and $\tilde{A}$ to their zero modes $A^{(0)} \equiv X/2\pi\alpha'$ and $\tilde{A}^{(0)} \equiv \tilde{X}/2\pi\alpha'$, where in the T-dual theory X and $\tilde{X}$ corresponds to collective coordinate of D0 and anti D0 brane [9]. As before the gauge fixing introduces the FP determinant [15]

$$\int db\, dc \exp(\mathrm{Tr} b\partial_t D_t c) = \prod_{i<j}\left(\frac{\sin[(x_i - x_j)R/2]}{(x_i - x_j)R/2}\right)^2 \left(\frac{\sin[(\tilde{x}_i - \tilde{x}_j)R/2]}{(\tilde{x}_i - \tilde{x}_j)R/2}\right)^2, \tag{3.2}$$

where x_i and $\tilde{x}_i$ are the eigenvalues of X and $\tilde{X}$ respectively. Now the denominator gets canceled with the respective Vandermonde determinant so that the usual measure factor $\Delta(x)^2\Delta(\tilde{x})^2$ is converted to the measure for unitary matrices

$$\prod_{i<j} \sin^2(\frac{(x_i - x_j)R}{2}) \sin^2(\frac{(\tilde{x}_i - \tilde{x}_j)R}{2}). \tag{3.3}$$

Note these are the measure for unitary matrices $U = e^{\frac{iXR}{2}}$, $\tilde{U} = e^{\frac{i\tilde{X}R}{2}}$.which are holonomy factors (we chose $\alpha' = 2$). Therefore the natural variables to be integrated over are the "holonomies" $U = e^{2i\pi A^{(0)}R}$, $\tilde{U} = e^{2\pi i\tilde{A}^{(0)}R}$. Once we gauge fix A and $\tilde{A}$ The partition function depends on the gauge field only through the global holonomy factor, given by the unitary matrix

$$\Omega = \hat{T} e^{i\int_0^{2\pi R} A(t)dt} \quad ; \quad \tilde{\Omega} = \hat{T} e^{i\int_0^{2\pi R} \tilde{A}(t)dt}. \tag{3.4}$$

In the $A = const$ gauge, in the path integral, the constant modes of A can be absorbed by redefining the fields $Z_-, \overline{Z}_-$ as

$$Z_-(t) \to e^{-iAt} Z_-(t) e^{i\tilde{A}t} \quad ; \quad \overline{Z}_-(t) \to e^{-i\tilde{A}t}\overline{Z}_-(t) e^{iAt}, \tag{3.5}$$

which replaces the periodic boundary condition

$$Z_\pm(2\pi R) = Z_\pm(0) \quad ; \quad \overline{Z}_\pm(2\pi R) = \overline{Z}_\pm(0)$$

by a $SU(N)$-twisted one [14], [17]

$$\begin{aligned} Z_+(2\pi R) = Z_\pm(0) \quad &; \quad \overline{Z}_+(2\pi R) = \overline{Z}_+(0) \\ Z_-(2\pi R) = \Omega Z_-(0)\tilde{\Omega}^{-1} \quad &; \quad \overline{Z}_-(2\pi R) = \tilde{\Omega}\overline{Z}_-(0)\Omega^{-1}, \end{aligned} \tag{3.6}$$

So in the constant A gauge integration with respect to the fields $Z_\pm(x), \overline{Z}_\pm(x)$ is Gaussian with the determinant of the quadratic form equal to one. Therefore it is reduced to the integral with respect to the initial values $Z_\pm, \overline{Z}_\pm = Z_\pm, \overline{Z}_\pm(0)$ of the action evaluated along the classical trajectories, which satisfy the twisted periodic boundary condition (3.6). Therefore the canonical partition function of the matrix model can be reformulated as an ordinary matrix integral with respect to the hermitian matrices Z_+ , Z_- ; $\overline{Z}_+$, $\overline{Z}_-$ and the unitary matrices $\Omega, \tilde{\Omega}$, :

$$\mathcal{Z}_N = \int dZ_+ dZ_- d\overline{Z}_+ d\overline{Z}_- d\Omega d\tilde{\Omega} e^{i\beta Tr(\overline{Z}_+ Z_- + Z_+\overline{Z}_- - q\overline{Z}_-\Omega Z_+\tilde{\Omega}^{-1} - qZ_-\tilde{\Omega}\overline{Z}_+\tilde{\Omega})}, \tag{3.7}$$

where we denote

$$q = e^{2i\pi R}. \tag{3.8}$$

Now note the above expression can be written as

$$\begin{aligned}
\mathcal{Z}_{\mathcal{N}} &= \int dZ_+ dZ_- d\overline{Z}_+ d\overline{Z}_- d\Omega d\tilde{\Omega} e^{i\beta \mathrm{Tr}(\overline{Z}_+ Z_- + Z_+ \overline{Z}_- - q\overline{Z}_+ \Omega Z_- \tilde{\Omega}^{-1} - qZ_+ \tilde{\Omega} \overline{Z}_- \Omega^{-1})} \\
&= \int dZ_+ dZ_- d\overline{Z}_+ d\overline{Z}_- d\Omega d\tilde{\Omega} e^{i\beta \mathrm{Tr}(\overline{Z}_+ Z_- + Z_+ \overline{Z}_- - q\overline{Z}_+ (\Omega Z_- \Omega^{-1}) \Omega \tilde{\Omega}^{-1} - qZ_+ \tilde{\Omega} \Omega^{-1} (\Omega \overline{Z}_- \Omega^{-1})} \\
&= \int dZ_+ dZ_- d\overline{Z}_+ d\overline{Z}_- d\Omega d\tilde{\Omega} e^{i\beta \mathrm{Tr}(\overline{Z}_+ Z_- + Z_+ \overline{Z}_- - q\Omega \tilde{\Omega}^{-1} \overline{Z}_+ (\Omega Z_- \Omega^{-1}) - qZ_+ \tilde{\Omega} \Omega^{-1} (\Omega \overline{Z}_- \Omega^{-1})} \\
&= \int dZ'_+ dZ_- d\overline{Z}'_+ d\overline{Z}_- d\Omega d\tilde{\Omega} e^{i\beta \mathrm{Tr}(\tilde{\Omega} \Omega^{-1} \overline{Z}'_+ Z_- + \overline{Z}_- Z'_+ \Omega \tilde{\Omega}^{-1} - q\overline{Z}'_+ \Omega Z_- \Omega^{-1} - qZ'_+ \Omega \overline{Z}_- \Omega^{-1}}) \\
&= \int dZ'_+ dZ_- d\overline{Z}'_+ d\overline{Z}_- d\Omega d\Omega' e^{i\beta \mathrm{Tr}(\Omega' \overline{Z}'_+ Z_- + \overline{Z}_- Z'_+ \Omega'^{-1} - q\overline{Z}'_+ \Omega Z_- \Omega^{-1} - qZ'_+ \Omega \overline{Z}_- \Omega^{-1}}),
\end{aligned} \tag{3.9}$$

where we define $Z_+ \tilde{\Omega} \Omega^{-1} = Z'_+ \quad ; \quad \Omega \tilde{\Omega}^{-1} \overline{Z}_+ = \overline{Z}'_+ \quad ; \quad \Omega' = \tilde{\Omega} \Omega^{-1}$.The last expression implies replacing $\tilde{X}$ by $\tilde{X} - X$, both running over the infinite real line. The redefinition of the variables will keep the measure invariant. So by generalizing Harishchandra-Itzykson-Zuber integral we can write the above partition function as [12], [13] [14].

$$\begin{aligned}
\mathcal{Z}_N(t) = \int_{-\infty}^{\infty} & \prod_{k=1}^{N} [dz_{+k}][dz_{-k}][d\overline{z}_{+k}][d\overline{z}_{-k}][d\Omega'_{kk}] \\
& [\det_{jk} \left(e^{i\Omega'_{jk} \overline{z}_{+j} z_{-k}} \right) \det_{jk} \left(e^{-iq\overline{z}_{+j} z_{-k}} \right) \det_{jk} \left(e^{i\Omega'^{-1}{}_{jk} \overline{z}_{-j} z_{+k}} \right) \\
& \det_{jk} \left(e^{-iq z_{+j} \overline{z}_{-k}} \right)],
\end{aligned} \tag{3.10}$$

where Ω'_{jk} which has only diagonal elements nonzero. Now we show that the grand canonical partition function can be written as a Fredholm determinant

$$Z(\mu, t) = \mathrm{Det}(1 + e^{-2\pi R\beta\mu} K_+ K_-), \tag{3.11}$$

where

$$\begin{aligned}
[K_+ f](\overline{z}_- z_-) &= \int [d\overline{z}_+][dz_+] dt e^{i(t\overline{z}_+ z_- + t^{-1} \overline{z}_- z_+)} f(\overline{z}_+ z_+), \\
[K_- f](\overline{z}_+ z_+) &= \int [d\overline{z}_-][dz_-] e^{-iq(\overline{z}_+ z_- + \overline{z}_- z_+)} f(\overline{z}_- z_-).
\end{aligned} \tag{3.12}$$

---

[12] $(\int dU e^{iTrUXU^{-1}Y} = Const. \frac{\det e^{ix_k y_l}}{\Delta(x)\Delta(y)}$where x_k and y_l are eigenvalues of X and Y and $\Delta(x)$ $\Delta(y)$ are Vandermonde determinant given by $\Delta(x) = \prod_{i \leq j} (x_k - x_l)$

[13] To express the part involving Ω' we used the fact that in the integral $(\int dU e^{i\mathrm{Tr}UXU^{-1}DY}$ where D is a complex diagonal matrix with $D^{-1} = D^\dagger$ and $Y = VyV^{-1}$ where y is the eigenvalue of Y and V is the unitary matrix diagonalizing Y. Now we can write $DY = V' d y V'^{-1}$ for some other diagonalizing matrix V' which exploits the fact that $\mathrm{Diag}((DY)^\dagger DY) = \mathrm{Diag}(Y^\dagger Y) = \mathrm{Diag}(Vy^* yV^\dagger)$ which implies the above expression (d is eigenvalue of D).So following the formal derivation of the integral we can write $(\int dU e^{iTrUXU^{-1}DY} = Const. \frac{\det e^{ix_k d_{kl} y_l}}{\Delta(x)\Delta(y)}$, which is nonzero only when $k = l$. for any diagonal matrix D.Note,we are not summing over k and l.Denominators gets canceled with the Vandermonde determinants appearing from $\overline{Z}_+, Z_+, \overline{Z}_-, Z_-$.

$$K_+K_-f(\overline{z}_+z_+) = \int [d\overline{z}_-][dz_-]dte^{i(t\overline{z}_+z_-+t^{-1}\overline{z}_-z_+)} \int [d\overline{z}_+'][dz_+']e^{-iq(\overline{z}_+'z_-+\overline{z}_-z_+')} f(\overline{z}_+'z_+'). \tag{3.13}$$

Note that t and t^{-1} denote the diagonal elements of Ω' and Ω'^{-1} corresponding to $\overline{z}_\pm, z_\pm$. Now note when we evaluate the determinant in a diagonalizable basis which is naturally given by $f(\overline{z}_\pm z_\pm)$, $K_+K_-f(\overline{z}_\pm z_\pm)$ will be independent of t i.e $\int dt$ will come out as an overall factor. So following the analysis of [14] the grand canonical partition function $\sum\limits_N e^{-2\pi R\beta N\mu}\mathcal{Z}_N$ can be expressed as the Fredholm determinant (3.11) which is same as that of $c = 1$ matrix model. Now following [24] we can express the partition function as $Trexp[-2\pi R\beta H]$. The gauge field A project the theory to singlet sector so that in the absence of perturbation, the grand canonical partition function is given by the Fredholm determinant

$$\mathcal{Z}(\mu) = \mathrm{Det}(1 + e^{-2\pi R\beta(\mu+H_0)}), \tag{3.14}$$

which must be same as (3.11). This can be interpreted as the grand canonical finite-temperature partition function for a system of non-interacting fermions in the inverse Gaussian potential. The Fredholm determinant can be computed once we know a complete set of eigenfunctions for the one-particle Hamiltonian H_o. Now in order to evaluate the free energy we need to find the density of states, it is conventional to introduce a cutoff Λ.

There is no momentum flow through the wall $\overline{z}z = x^2 + y^2 = |\Lambda|^2$ is implied by the condition $(\hat{x}\hat{p}_y + \hat{x}\hat{p}_y)\psi_\pm(x, y)|_{(x^2+y^2=|\Lambda|^2)} = (\hat{\overline{z}}_+\hat{z}_+ - \hat{\overline{z}}_-\hat{z}_-)\psi_\pm(\overline{z}, z)|_{(\overline{z}z=|\Lambda|^2)} = 0$, which has a solution

$$\psi^E_+(\Lambda) = \psi^E_-(\Lambda). \tag{3.15}$$

This condition is satisfied for a discrete set of energies $E_n (n \in Z)$ defined by

$$\phi_0(E_n) - E_n \log\Lambda + 2\pi n = 0. \tag{3.16}$$

From (3.16) we can find the density of the energy levels in the confined system

$$\rho(E) = \frac{\log\Lambda}{2\pi\beta} - \frac{1}{2\pi\beta}\frac{d\phi_0(E)}{dE}, \tag{3.17}$$

as derived in [14] Now we can calculate free energy $\mathcal{F}(\mu, R) = \log\mathcal{Z}(\mu, R)$ as

$$\mathcal{F}(\mu, R) = \int_{-\infty}^{\infty} dE\, \rho(E) \log\left[1 + e^{-2\pi R\beta(\mu+E)}\right], \tag{3.18}$$

with the density (3.17). Integrating by parts in and dropping out the Λ-dependent piece, we get

$$\mathcal{F}(\mu, R) = -\frac{1}{2\pi\beta}\int d\phi_0(E) \log\left(1 + e^{-2\pi R\beta(\mu+E)}\right) = -R\int_{-\infty}^{\infty} dE \frac{\phi_0(E)}{1 + e^{2\pi R\beta(\mu+E)}}.. \tag{3.19}$$

We close the contour of integration in the upper half plane and take the integral as a sum of residues. This gives for the free energy

$$\mathcal{F} = -i \sum_{r=n+\frac{1}{2}>0} \phi_o\left(ir/R - \mu\right). \tag{3.20}$$

As the Fredholm determinant is similar to that of $c = 1$ MQM so following the analysis of [14], [15] we can see From (3.20) it follows that [19]

$$2\sin\frac{\partial_\mu}{2\beta R} \cdot \mathcal{F}(\mu) = \phi_o(-\mu). \tag{3.21}$$

Also its shown that the free energy can be expressed as

$$\mathcal{F}_{\text{pert}}(\mu)_{\{t_k=0\}} = -\frac{R}{2}\mu^2 \log\frac{\mu}{\Lambda} - \frac{R+\frac{1}{R}}{24}\log\frac{\mu}{\Lambda} + R\sum_{h=2}^{\infty}\mu^{2-2h}c_h(R), \tag{3.22}$$

3.2 Free energy of Type 0A matrix model on a circle with D brane

In this section we will consider the type 0A matrix model path integral in the presence of a D brane and show the grand canonical partition function can be expressed as the Fredholm determinant. We consider the brane in the NS NS sector and show that how to generalize the analysis for the brane in any other sector. Consider the path integral in the presence of the macroscopic loop operator localized at t_o, which is the generalization of (3.1). The classical action will remain periodic even in the presence of D brane so we can express the

$$\int dZ_+ dZ_- d\overline{Z}_+ d\overline{Z}_- dA d\tilde{A} e^{-\beta\int_0^{2\pi R} dt \mathrm{Tr}\left[\overline{Z}_+ D_A Z_- + Z_+ D_A \overline{Z}_- + \frac{1}{2}(\overline{Z}_- Z_+ + \overline{Z}_+ Z_-)\right] + \mathrm{Tr}W(t_o)}, \tag{3.23}$$

The macroscopic loop operator depends on diagonal elements only, so the partition function (3.7) can be expressed as

$$\mathcal{Z}_N(t) = \int_{-\infty}^{\infty}\prod_{k=1}^{N} [dz_{+k}][dz_{-k}][d\overline{z}_{+k}][d\overline{z}_{-k}][dt_k]\det{}_{jk}\left(e^{it_{jk}^{-1}z_{-j}\overline{z}_{+k}}\right)\det{}_{jk}\left(e^{-iqz_{-j}\overline{z}_{+k}}\right)\det{}_{jk}\left(e^{it_{jk}\overline{z}_{-j}z_{+k}}\right)$$
$$\det{}_{jk}\left(e^{-iq\overline{z}_{-j}z_{+k}}\right) \quad exp[\sum_i log(1 + \frac{\overline{z}_{+i}z_{+i} + \overline{z}_{-i}z_{-i} + \overline{z}_{+i}z_{-i} + \overline{z}_{-i}z_{+i}}{\mu_B^2})], \tag{3.24}$$

(when we have off-diagonal t is zero , also we are not summing over j,k)
where $\sum_i$ is coming from Trace and again above can be expressed as

$$\mathcal{Z}_N(t) = \int_{-\infty}^{\infty}\prod_{k=1}^{N}[dz_{+k}][dz_{-k}][d\overline{z}_{+k}][d\overline{z}_{-k}]\det{}_{jk}\left(e^{it_{jk}^{-1}z_{-j}\overline{z}_{+k}}\right)\det{}_{jk}\left(e^{-iqz_{-j}\overline{z}_{+k}}\right)\det{}_{jk}\left(e^{it_{jk}\overline{z}_{-j}z_{+k}}\right)$$
$$\det{}_{jk}\left(e^{-iq\overline{z}_{-j}z_{+k}}\right)\prod_{r=1}^{N}(1 + \frac{\overline{z}_{+r}z_{+r} + \overline{z}_{-r}z_{-r} + \overline{z}_{+r}z_{-r} + \overline{z}_{-r}z_{+r}}{\mu_B^2}). \tag{3.25}$$

Now in order to write the above expression we have used the fact that the classical action is periodic even in the presence of the macroscopic loop operator $Z(2\pi R) = Z(0)$. However at the quantum level there is a discontinuity of state $|\psi(2\pi R-\epsilon)\rangle \neq |\psi(0)+\epsilon\rangle$. This causes the absence of the vortex d.o.f. Now if $f(\overline{z}_\pm z_\pm)$ is the function which form the representation of K_+ and K_- (3.12, 3.13), the action of W on f is given by

$$\hat{W}f(\overline{z}_\pm z_\pm) = (1 + \frac{2\hat{\overline{z}}_\pm \hat{z}_\pm + \hat{\overline{z}}_+\hat{z}_- + \hat{\overline{z}}_-\hat{z}_+}{\mu_B^2})f(\overline{z}_\pm z_\pm). \tag{3.26}$$

Now, the operator $\hat{W}$ does not introduce any interaction between the fermions, so no off diagonal terms from W. Now from (3.11) and (3.24) we can write the grand canonical partition $\sum_N e^{-2\pi R\beta N}\mathcal{Z}_N$ as

$$\mathcal{Z}(\mu) = \det(1 + e^{-2\pi R\beta\mu}WK). \tag{3.27}$$

Now in order to evaluate (3.27) following (3.13) a representation of K is formed by the basis $f(\overline{z}_{+i}z_{+i})$ with i runs from 1 to N. Also from (3.13) it follows that $f(\overline{z}_+z_+) \sim (\overline{z}_+z_+)^n$. So when we evaluate the expectation value of WK in this basis, in the expression of W we see that $\langle\overline{z}_\pm z_\pm\rangle = 0$ as on a closed contour the angular integral will vanish. in the inner product, the other term $\overline{z}_+z_- + z_+\overline{z}_-$ expresses nothing but the hamiltonian of which f is an eigenfunction. So if ψ_n are the set of functions which diagonalizes K we can write (3.27) as

$$\sum_n \log\langle\psi_n|(1 + e^{-2\pi R\beta\mu}\hat{W}\hat{K})|\psi_n\rangle, \tag{3.28}$$

where

$$\hat{W}K_+K_-f(\overline{z}_+z_+) = \int [d\overline{z}_-][dz_-]e^{i(\overline{z}_+z_- + \overline{z}_-z_+)}[d\overline{z}_+{}'][dz_+{}']e^{-iq(\overline{z}_+{}'z_- + \overline{z}_-z_+{}')}$$
$$(1 + \frac{\overline{z}_+{}'z_- + \overline{z}_-z_+{}'}{\mu_B^2})f(\overline{z}_+{}'z_+{}'). \tag{3.29}$$

As the expression depends on $(\overline{z}_+z_- + \overline{z}_-z_+)$ which is the expression for free hamiltonian H_o so comparison with (3.14), Fredholm determinant is expected to be given by

$$\mathcal{Z}(\mu) = \det(1 + e^{-2\pi R\beta(\mu+H_0) - \log(1-\frac{2H_o}{\mu_B^2})}). \tag{3.30}$$

This is,we are going to analyze in the next part of this section.

3.3 Evaluation of the thermal partition function

In this section we are going to study type 0A MQM in the presence of D brane with time t compactified on a circle, evaluate and analyze the free energy. In the absence of the brane when we compactify string theory on a circle of radius R, in the dual MQM the Schrodinger equation have periodic solution i.e $\psi(t) = \psi(t+2\pi R)$, which implies

$E = \frac{n}{R}$. Now consider the theory with D brane which can be accomplished by including a macroscopic loop operator localized at $t = t_o = 0 \equiv 2\pi R$ (say) to the action. From previous discussion it follows that in the presence of the operator Schrodinger equation will have well defined solution only in the region $0 \leq t \leq 2\pi R$ when discontinuity occur at the respective point and we have $\psi(2\pi R - \epsilon) \neq \psi(2\pi R + \epsilon)$ in the limit $\epsilon \to 0$. This is consistent with the fact that the presence of a spacelike brane breaks the winding symmetry and apparently the theory correspond to that of an open string. At the end we will see how the closed string scenario arise in this picture. Now in a compact time we must have the condition $\psi(t) = \psi(2\pi R + t)$. So effectively we can view the theory as MQM on a line of length $2\pi R$ with two δ–potential along with the operator $\hat{W}$ (where one is the image of the other, superimposed) placed at its two ends. When we cross the boundary on either side, situation repeats [14], i.e we can define the theory on any of the slices $2\pi(n-1)R \leq t \leq 2\pi nR$. So effectively we have the time dependent Schrodinger equation with double delta potential well as:

$$
\begin{aligned}
[\, i\frac{\partial}{\partial t} - \{\delta(t) + \delta(t - 2\pi R)\} W(\hat{\overline{z}}_\pm \hat{z}_\pm(t), H_o)] \Psi(\overline{z}_\pm z_\pm, t) \\
= \mp i \left[z_\pm \frac{\partial}{\partial z_\pm} + \overline{z}_\pm \frac{\partial}{\partial \overline{z}_\pm} + 1 \right] \Psi(\overline{z}_\pm z_\pm, t), \qquad (3.31)
\end{aligned}
$$

We have the discontinuity

$$
\begin{aligned}
\psi(\epsilon) - \psi(-\epsilon) &= \hat{W} \psi_o(t = 0) \\
\psi(2\pi R + \epsilon) - \psi(2\pi R - \epsilon) &= \hat{W} \psi_o(t = 2\pi R). \qquad (3.32)
\end{aligned}
$$

In order to evaluate the partition function we must need to know what is the right Hilbert space describe the wave function ψ on the circle. This is because we know that the Hilbert space $\{|E\rangle\}$ and $\{|E \pm ni\rangle\}$ cannot be mapped to each other. So, to answer this note that when we define the Schrodinger equation in the double delta potential well in an uncompactified direction we have one type of the solution inside the well, while the solution at the left and the rightside of the well differs from the same, decided by the discontinuity (3.32). Now compactification on a circle of length $2\pi R$ imply the outside region of the well is just squeezed to a point $t = 0 \equiv 2\pi R$ and the wave function at the left and right side of the double delta-well are given by the wave function at the right and left ϵ–neighborhood of that point. Now in uncompactified time we had free fermion wave function (2.28) in the region $-\infty$ to t_o while from t_o to ∞ the wave function is

[14] note it never implies periodicity, its just similar to the situation of an open string in 2D with Dirichlet boundary condition in compact direction and identification of the matter direction with t. It winds along the circle m times although ends are not identified. The open string which wraps m times a circle of length $2\pi R'$ with $2\pi m R' = 2\pi R$, we can define same theory on either of the slices $2\pi(n-1)R \leq t \leq 2\pi nR$, crossing the boundary of the slice implies going back from that end of the string to the other and hence the situation repeats

described by (2.44). So in the compactified time we have the ambiguity that which one should describe the fermionic wave function in the region $0 \leq t \leq 2\pi R$. To resolve recall the macroscopic loop operator $\hat{W}$ (2.15) and the constraint (2.25) are symmetric under $\overline{z}_+, z_+ \to \overline{z}_-, z_-$. So in the Schrodinger equation(2.12) we have the symmetry,

$$t \to 2\pi R - t; \quad \hat{W} \to -\hat{W}, \tag{3.33}$$

which takes an wave function $\overline{z}_+, z_+ \to \overline{z}_-, z_-$ representation[15] along with reversal of the sign of the energy $E \to -E$ in the free fermionic wave function as introduced in (2.28). As both $\overline{z}_+, z_+$ and $\overline{z}_-, z_-$ describes same wave function in different representation so it must be a symmetry inside the double delta well (note this is never a symmetry outside the well where the fermion can see only one potential barrier). The condition (2.25) remains unaffected by this symmetry and we can project the wave function $t = t_o = 0 \equiv 2\pi R$ to the physical sector. Now under the transformation (3.33) the free fermion wave function (2.28) is just changed by a phase $e^{2iE\pi R}$ whereas according to (2.28,2.30,2.41) the wave function describes a state $|E \pm ni\rangle$ goes from $e^{-iEt \pm nt} \to e^{iE(2\pi R - t) \mp (2\pi nR - nt)}$. Although the transformation (3.33) take the wave function from z_+ to z_- representation but the both have the same time dependent part. So when we consider the wave function (2.41) we see it does not respect the symmetry. Note unlike the closed string momentum modes which respects the symmetry in Euclidean time because of their periodicity on the circle, $W(\hat{\overline{z}}_\pm \hat{z}_\pm, H_o)$ generates discrete shift in energy $E \pm ni$ at any R, which leads to the violation of symmetry. So we conclude the wave function inside the well which is our compact time $0 \leq t \leq 2\pi R$ must corresponds to that of a free fermion (2.28). We will come to the string theoretical interpretation in the next subsection. The wave function at $t = 0 \equiv 2\pi R$ corresponds to (2.41) so the partition function corresponds to the transition amplitude

$$\begin{aligned}
\lim_{\epsilon \to 0} \langle \psi_o(\epsilon) | \psi_>(2\pi R - \epsilon) \rangle &= \langle \psi_o [e^{-\beta \int_0^{2\pi R} dt \hat{H}_o} e^{\hat{W}(t_o)}] \psi_o \rangle \\
&= \mathrm{Tr}_{\psi_o^E} [e^{-2\pi R \beta \hat{H}_o} [e^{\hat{W}(t_o)}] \;] \\
&= \mathrm{Tr}_{\psi_o^E} \left[e^{-2\pi R \beta [\hat{H}_o - \frac{1}{2\pi R \beta} \hat{W}(t_o)]} \right],
\end{aligned} \tag{3.34}$$

where $\mathrm{Tr}_{\psi_o^E}$ implies the summation over all free fermion eigenfunctions and $\psi_>$ corresponds to (2.41). As $\beta \to \infty$ at double scaling limit, so inside the partition function we can replace it by $\psi_> \to e^{-W(t_o)} \psi_o$ [16]. Note when we consider Schrodinger equation, the contribution

[15] this is from the definition of $\overline{z}_\pm, z_\pm$ (2.6) and the reversal of the sign of W is explained from (3.32),the transformation changes the sign in the r.h.s of (3.32) because the time interval $\epsilon \to -\epsilon$ under the transformation and hence the relation will remain unchanged.Note this transformation is also associated with the reversal of the sign of the gauge field A. but we chose axial gauge $a = \tilde{a}$,where $a, \tilde{a}$ are the zero modes of gauge fields $A, \tilde{A}$, so this is not affected

[16] In order to reach from the 1rst to 2nd step in (3.34)we utilize the fact that we can scale the time $t \to \beta t$ so that the term with the macroscopic loop operator $\int dt W(t) \delta(t - t_o)$ will get a factor $\frac{1}{\beta}$.

from the term $\frac{1}{2\pi R\beta}\hat{W}(t_o)$ in the expression of hamiltonian (3.34) at double scaling limit will not be negligible due to a transformation of variable which leads to the physics at the vicinity of the top of the potential, as we discussed section 2.1 and can be found [1]. From (2.15) we know that in a single fermionic state the presence of D brane implies implies the insertion of the following operator

$$e^{\hat{W}(t_o)} = 1 + \frac{\hat{\overline{z}}_+\hat{z}_+ + \hat{\overline{z}}_-\hat{z}_- + \hat{\overline{z}}_-\hat{z}_+ + \hat{\overline{z}}_+\hat{z}_-}{\mu_B^2}. \tag{3.35}$$

Here first we will evaluate the partition function for single fermionic d.o.f in order to understand the behaviour of the system in the presence of a brane. Next we will derive the grand canonical partition function.

Now following the discussion in section 2 we can represent the wave function (2.40) at $t = 2\pi R - \epsilon$ in (3.34) with the operator $\hat{W}$ expressed either in terms of $\hat{\overline{z}}_+, \hat{z}_+$ or $\hat{\overline{z}}_-, \hat{z}_-$ representation. We have shown in the Appendix that $\langle \overline{z}_+ z_+|\hat{\overline{z}}_+\hat{z}_+|\overline{z}_+ z_+\rangle$ and $\langle \overline{z}_- z_-|\hat{\overline{z}}_-\hat{z}_-|\overline{z}_- z_-\rangle$ diverge. So we must express $\hat{W}$ as $W(\hat{\overline{z}}_-\hat{z}_-, H_o)$ for the basis $|\overline{z}_+ z_+, E\rangle$ basis and vice versa. So applying (A.7) we can write (3.34) as:

$$\mathrm{Tr}_{\psi_o}\left[e^{-2\pi R\beta[\hat{H}_o - \frac{1}{2\pi R\beta}\log(1-\frac{2\hat{H}_o}{\mu_B^2})]}\right] = \mathrm{Tr}_{\psi_o}\left[e^{-2\pi R\beta H'_o}\right], \tag{3.36}$$

Where

$$\hat{H}'_o = \hat{H}_o - \frac{1}{2\pi R\beta}\log(1-\frac{2\hat{H}_o}{\mu_B^2}) \quad ; \quad E'_o = E - \frac{1}{2\pi R}\log(1-\frac{2E}{\mu_B^2}), \tag{3.37}$$

with E'_o is the eigenvalue of H'_o(note that we omitted β factor from the expression of E'_o following the discussion of section 2, which can be done at double scaling limit by redefinition of the variable). Before any further analysis let us make the comment that here we have considered the macroscopic loop operator in the NS NS sector. For any other sector we can use analysis of section 2, expressing W in terms of $\hat{\overline{z}}_-.\hat{z}_-(\hat{\overline{z}}_+, \hat{z}_+)$ in $\overline{z}_+, z_+(\overline{z}_-, z_-)$ representation and using (A.7) to express $W(t_o)$ complete in terms of H_o within the trace. Although we will have a very different expression of H'_o but the analysis will remain same. Also if we did not apply this condition (2.25), note we will have the term $\langle\hat{\overline{z}}_-\hat{z}_-\rangle_-$($\langle\hat{\overline{z}}_+\hat{z}_+\rangle_+$) in the partition function from the expression of $\hat{W}$. This gives rise to an infinite contribution to the partition function $\lim_{r\to\infty} re^{i\phi(E-i)}$ (where ϕ is the phase of the wave function) and so the partition function diverge. This is the signature of the presence of an unphysical degree of freedom leads to instability of the system due to leakage. Now according to the discussion of section 2, at the double scaling limit the partition function (3.36) can be expressed as the sum over $E'_o$the eigenvalue of H'_o as:

$$\mathrm{Tr}_{\psi_o}\left[e^{-2\pi R\beta H'_o}\right] = \sum_E \left[e^{-2\pi R\beta[E - \frac{1}{2\pi R}\log(1-\frac{2E}{\mu_B^2})]}\right] \tag{3.38}$$

Hence in the double scaling limit where $\beta \to \infty$ and with Euclidean time, we can lift up the term to the exponential and the exponent gives an exact expression what we have obtained from the path integral (3.23) . Also following the discussion of section 2 we can directly add $W(t_o)$ in the expression of hamiltonian in Euclidean time to get the expression (3.34)

Now note that E'_o,the eigenvalue of H'_o, has branch cut at $E = \frac{\mu_B^2}{2}$ so we need to subtract a small cut-off $\log\epsilon$ in order to have an well defined expression of the energy and after subtraction $E \to E'$ is an one to one mapping, Also note at the singular point, $E = \frac{1}{2}\mu_B^2$, $e^{-2\pi R\beta E'}$ is trivially zero and so it will not contribute any pole to integrand. Now the string theory compactified at time interval $2\pi R$ is described by the grand canonical partition function of fermion at finite temperature $\frac{1}{2\pi R}$ and chemical potential μ. So the free energy $\mathcal{F} = \log\mathcal{Z}$ in the presence of Dbrane is given by

$$\mathcal{F}(\mu) = \int_{\infty}^{\infty} dE\rho(E)log(1 + e^{-\beta(\mu+E'(E))}), \tag{3.39}$$

where$\rho(E)$ is given in (3.17)

$$e^{i\phi_0(E)} = R(E) = \frac{\Gamma(iE + 1/2)}{\Gamma(-iE + 1/2)}. \tag{3.40}$$

Now we can calculate free energy $\mathcal{F}(\mu, R) = \log\mathcal{Z}(\mu, R)$ as.

$$\mathcal{F}(\mu, R) = \int_{-\infty}^{\infty} dE\rho(E)\log[1 + e^{-\beta(\mu+E'(E))}]. \tag{3.41}$$

with the density (3.17),and from (3.39,3.37) the free energy is given by

$$\begin{aligned}
\mathcal{F}(\mu, R) &= -\frac{1}{2\pi}\int d\phi_0(E)\log\left(1 + e^{-2\pi R\beta(\mu+E'(E))}\right) \\
&= -R\int_{-\infty}^{\infty} dE'\frac{\phi_0(E(E'))}{1 + e^{-2\pi R\beta(\mu+E'(E)))}} \\
&= -i\sum_{r=n+\frac{1}{2}>0} \phi_o(E(E' = \frac{ir}{\beta R} - \mu)) \\
&= -i\sum_{r=n+\frac{1}{2}>0} \phi'_o(\frac{ir}{R} - \mu),
\end{aligned} \tag{3.42}$$

where we have

$$\phi'_0(E') = \phi_0(E) \tag{3.43}$$

So from the expression of the density of the energy eigenstates (3.17) (ignore the Λ factor) this implies the number of energy eigenstates between E to E+dE is same as that of between E' to $E' + dE'$. So the partition function on a circle in the presence of D brane corresponds to a deformation in static Fermi sea where the deformation is expressed as $E = -\mu \Rightarrow E' = -\mu$ with all the energy eigenstates are in one to one mapping. Note that the partition function is getting contribution only from the deformation of Fermi surface instead of excitation modes. This is because we are in compact dimension and its only the trace over energy eigenstates contribute, which we will explain more in the next subsection. Finally the expression of the free energy $\mathcal{F}(\mu, R)$ in (3.42) along with (3.43) suggests the following

$$\mathrm{Tr}_{\psi_o}\left[e^{-2\pi R\beta[\hat{H}_o - \frac{1}{2\pi R\beta}\hat{W}(t_o)]}\right] = \mathrm{Tr}_{\psi_o}[e^{-2\pi R\beta\{H_o - \frac{1}{2\pi R\beta}\log(1-\frac{2H_o}{\mu_B^2})\}}]$$

$$= \mathrm{Tr}_{\psi_o}[e^{-2\pi R\beta\{H_o - \frac{1}{2\pi R\beta}f(H_o)\}}]$$
$$= \mathrm{Tr}_{\psi_o}[e^{-2\pi R\beta(H'_o)}]$$
$$= \mathrm{Tr}_{\psi'_o}[e^{-2\pi R\beta(H_o)}], \tag{3.44}$$

where

$$f(H_o) = \log(1 - \frac{2H_o}{\mu_B^2}) \quad ; \quad H'_o = H_o - \frac{1}{2\pi R\beta}f(H_o). \tag{3.45}$$

and in the last step we made a transformation from the basis $\psi_o^{\pm}(E) = e^{\mp i\phi_o(E)}e^{-iEt}(\overline{z}_{\pm}z_{\pm})^{\pm iE-\frac{1}{2}} \to \psi_o^{\pm}(E') = e^{\mp i\phi'_o(E')}e^{-iE't}(\overline{z}_{\pm}z_{\pm})^{\pm iE'-\frac{1}{2}}$ with $E' = E - \frac{1}{2\pi R}\log(1 - \frac{2E}{\mu_B^2})$ and also $\phi'_o(E') = \phi_o(E)$. Note as we discussed in section 2, although the contribution from the macroscopic loop operator has a factor $\frac{1}{\beta}$ however in the double scaled hamiltonian it cannot be ignored and the shifted energy will be given by E'. Above expression implies that the Type 0A MQM on a circle in the presence of a D brane can be viewed as a free theory with the free hamiltonian H_o with the wave function replaced by the above one. This point will be relevant in section 5. Note the relation (3.21,3.22) can be expressed as

$$2\sin\frac{\partial_\mu}{2\beta R} \cdot \mathcal{F}(\mu) = \phi'_o(-\mu). \tag{3.46}$$

Also note that from nonlinear relation between the effective hamiltonian H'_o and the free hamiltonian H_o its evident that in the genus expansion of free energy in the relation (3.22) will have both odd and even powers of μ(this is because the new free energy corresponds to replacing $\mu \to E(E')|_{E'=\mu}$ which follows from (3.39)). This is the signature of the presence of surface with boundary which is the implication from MQM/string theory duality.

3.4 String theoretical interpretation

Let us very briefly say about the string theory side of the above story. The matrix model partition function (3.44) corresponds to the disk amplitude

$$\langle B|q^{(L_o+\overline{L}_o)}|I\rangle \tag{3.47}$$

Where $|B\rangle$ stands for boundary state and the above expression resembles (3.34) on the matrix model side. The invariance of theory under the symmetry (3.33) is related to the symmetry of the boundary state (2.47,2.48) under interchange of the left and right moving tachyonic components in W

$$U(k)e^{ik(X+\phi)-\sqrt{2}\phi} \leftrightarrow U(-k)e^{ik(X-\phi)-\sqrt{2}\phi} \tag{3.48}$$

as well as other Virasoro primaries obtained from discrete shift of matter and Liouville momentum of tachyonic modes. One can see that in absence of D brane this is a symmetry

in MQM. The symmetry is implemented from the interchange $X \to 2\pi R - X$ followed by a sign reversal of momentum $k \to -k$ which is in accordance (3.33) with $k = \frac{n}{R}$ and $X \to it$, U(k) is the wave function. This keeps the matter part (temporal part in matrix model) of the operator/state invariant. The reason for this symmetry in matrix model is that the tachyonic states obtained from the 2nd quantized free fermionic theory (ground state) or the collective field theory are actually in one to one to one correspondence with the above operators(3.48), as we explained in section 2.3. Consequently we found that the symmetry (3.33) projects the Hilbert space between $0 < t < 2\pi R$ to free fermionic ground state. This is because the discrete tachyonic modes arises from Minkowskian theory with shift $X \to iX$; $k \to ik \ \Rightarrow e^{ikX}(e^{ikt}) \to e^{ikX}(e^{ikt})$. However the states created by the action of the operator $(\hat{\overline{z}}_{-}\hat{\overline{z}}_{+})^{n}(\hat{\overline{z}}_{-}\hat{\overline{z}}_{+})^{m}$ on ground state, although describes a state with imaginary energy but remains in Minkowskian time (unperiodic) and so projected out by the symmetry within $0 < t < 2\pi R$. However these states arise on expansion of $W(\hat{\overline{z}}_{-}\hat{z}_{-}, \hat{\overline{z}}_{+}\hat{z}_{+}, H_o)$ which actually causes the excitation of a fermion over free fermionic ground state,creates a coherent state.

So we have only free fermionic ground state in the region $0 < t < 2\pi R$ which in string theory indicates that we have only free closed string modes in the region $0 < X < 2\pi R$. Coherent states are strongly localized at the point of insertion of $\hat{W}$. In matrix model partition function, the absence of the excitation modes has an explanation in the fact that on a circle in the presence of a brane, partition function corresponds to transition $\psi \to \hat{W}\psi$ exactly at $t = t_o = 0 \equiv 2\pi R$, where $|\psi\rangle$ is the free fermionic ground state. So the operators from W which generates excitation naturally will not contribute in the trace. This, alongwith the condition (2.25)(which ensures the conservation of fermion number) implies that the partition function will corresponds to that of a deformed Fermi surface. The poles of the partition function (3.42) corresponds to free closed string tachyons in the bulk. In section 5 we will see that same features will be reflected even when we turn on the tachyonic background.

4 Fermionic scattering and semiclassical analysis

In this section we will study scattering of fermions in the presence of D brane and tachyonic background at quasiclassical limit. The scattering amplitude is given by

$$S = \langle \beta, t \to \infty | \alpha, t \to -\infty \rangle, \tag{4.1}$$

where α and β denotes the incoming and outgoing state. As the single incoming and outgoing state is given by $|\overline{z}_{+}z_{+}\rangle$ and $|\overline{z}_{-}z_{-}\rangle$ respectively [28], so

$$S = \langle \overline{z}_{-}z_{-}, \text{out} | \overline{z}_{+}z_{+}, \text{in} \rangle. \tag{4.2}$$

Now note that $\overline{z}_{+}z_{+}$ and $\overline{z}_{-}z_{-}$ representations are related by a unitary operator $\hat{S}$, which in our case is nothing but the Fourier transformation on the complex plane. Recall, the

energy eigenstates in absence of the D brane are given by (2.28). The wave functions in $(z_+, \bar{z}_+)$ and $(z_-, \bar{z}_-)$ representations are related by

$$\begin{aligned}\psi_-(z_-, \bar{z}_-) &= \hat{S}\psi_+(z_-, \bar{z}_-) \\ &= \int dz_+ d\bar{z}_+ K(\bar{z}_-, z_+) K(z_-, \bar{z}_+) \psi_+(z_+, \bar{z}_+),\end{aligned} \tag{4.3}$$

where $K(z_-, z_+) = \frac{1}{\sqrt{2\pi}} e^{iz_- z_+}$. Acting on energy eigenstates, we have

$$\hat{S}\psi_+^E = \mathcal{R}(E)\psi_-^E, \quad \mathcal{R}(E) = \frac{\Gamma(iE + \frac{1}{2})}{\Gamma(-iE + \frac{1}{2})}. \tag{4.4}$$

The factor $\mathcal{R}(E)$ is a pure phase

$$\overline{\mathcal{R}(E)}\mathcal{R}(E) = \mathcal{R}(-E)\mathcal{R}(E) = 1, \tag{4.5}$$

which proves the unitarity of the operator $\hat{S}$. Now in absence of the D brane wave function (2.28) evolve according to free hamiltonian H_o so from orthonormality of the wave functions (4.2) is given by $\mathcal{R}(E)\delta(E_+ - E_-)$. The operator $\hat{S}$ relates the incoming and the outgoing waves and therefore can be interpreted as the fermionic scattering matrix. The factor $\mathcal{R}(E)$ is identical to the the reflection coefficient. This condition can also be expressed as the orthonormality of in and out eigenfunctions

$$\langle \Psi_-^{E_-} | K | \Psi_+^{E_+} \rangle = \delta(E_+ - E_-), \tag{4.6}$$

with respect to the scalar product. We usually absorb the factor$\mathcal{R}(E)$ in phase by defining

$$e^{i\phi(E)} = \mathcal{R}(E) \tag{4.7}$$

to make the wave function biorthogonal, where $\phi(E)$ is the phase of the incoming and the outgoing wave function(2.28). Now consider the presence of D brane. For a single fermionic state, (4.2) is given by

$$\langle \overline{z}_+ z_+, t = \infty | e^{\hat{W}(t)} | \overline{z}_- z_-, t = \infty \rangle = \langle \overline{z}_+ z_+, \mathrm{out} | (1 - \frac{\overline{z}_+ z_+ + \overline{z}_- z_- - 2H_o}{\mu_B^2}) | \overline{z}_- z_-, \mathrm{in} \rangle. \tag{4.8}$$

Now according to (2.32) W will be expressed either in $\overline{z}_+ z_+$ or $\overline{z}_- z_-$ mode

$$\begin{aligned}\langle z_+, E_+, \mathrm{out} | e^{\hat{W}(t_o)} | z_-, E_-, \mathrm{in} \rangle &= \langle z_+, E_+ | (1 + \frac{2\overline{z}_- z_- - 2H_o}{\mu_B^2})_{t=t_o} | z_-, E_- \mathrm{in} \rangle \\ &= \langle z_+, E_+ | (1 + \frac{-2H_o}{\mu_B^2}) | z_-, E_- \rangle \\ &= \langle z_+, E_+ | (1 - \frac{2E}{\mu_B^2}) | z_-, E_- \rangle \\ &= \mathcal{R}(E_+) e^{log(1 - \frac{2E}{\mu_B^2})} \delta(E_+ - E_-),\end{aligned} \tag{4.9}$$

where in the 2nd step we have $\langle \overline{z}_+ z_+, E_+; \text{out}|\hat{\overline{z}}_- \hat{z}_-|\overline{z}_- z_-, E_-, \text{in}\rangle = 0$ from (A.6) . Now the presence of the D brane will modify the phase of the outgoing state over the incoming, which is given by the factor $e^{log(1-\frac{2E}{\mu_B^2})}$. So for the change of phase $\delta\phi(E)$ we can write

$$e^{-i\frac{\delta\phi(E)}{2}} = e^{log(1-\frac{2E}{\mu_B^2})} \tag{4.10}$$

Now in (4.10) using the relation (4.7) will leave us with the amplitude $e^{-i\frac{\delta\phi(E)}{2}}$. In [43] its explained that the complex phase in the wave function is the signature of tunneling and we can presume the above factor accounts for the same. Also note that instead of the above macroscopic loop operator in NS sector, if we took the macroscopic loop operator in some other sector given by $W'(\frac{\overline{z}_+ z_+, \overline{z}_- z_-, H_o}{\mu_B^2})$ according to the discussion in Appendix we will have the phase shift of the outgoing state $\frac{\phi(E)}{2} + iW'(1 - \frac{2E}{\mu_B^2})$.

Lets consider scattering of a tachyonic state (which are being created from the action of $(\hat{\overline{z}}_\pm \hat{z}_\pm)^n$ on fermionic ground state) from D brane. This is better understood from the collective field theory where scattering to a single tachyonic state with energy E is given by $\sim U(E)$ where U(E) is given by (2.48). This supports the fact that the D brane act as a coherent source of closed strings. Now we consider the classical limit $\beta \to \infty$. At this limit the ground state of MQM is obtained by filling all energy levels up to some fixed Fermi energy which we choose to be $E_F = -\mu$. Quasiclassically every energy level corresponds to a certain trajectory in the phase space of $\overline{z}_+ z_+, \overline{z}_- z_-$ variables and they are separated by a factor $\frac{1}{\beta}$. The Fermi sea can be viewed as a stack of all classical trajectories with $E \leq E_F$ and the ground state is completely characterized by the curve representing the trajectory of the fermion with highest energy E_F. For the Hamiltonian H_o all trajectories are hyperboles $\overline{z}_+ z_- + \overline{z}_- z_+ = -E$ and the profile of the Fermi sea is given by

$$\overline{z}_+ z_- + \overline{z}_- z_+ = -\mu. \tag{4.11}$$

First consider the theory without D brane. Then the low lying collective excitations are represented by deformations of the Fermi surface,

$$\overline{z}_+ z_- + \overline{z}_- z_+ = M(\overline{z}_+ z_- + \overline{z}_- z_+). \tag{4.12}$$

In order to study the scattering with such deformed background we will follow the analysis of [16]. The perturbed wave functions are related to the old ones by a phase factor

$$\psi^E_\pm(\overline{z}_\pm z_\pm) = e^{\mp i\varphi_\pm(\overline{z}_\pm z_\pm;E)} \psi^E_\pm(\overline{z}_\pm z_\pm), \tag{4.13}$$

whose asymptotics at large $\overline{z}_\pm z_\pm$ characterizes the incoming/outgoing tachyon state. We split the phase into three terms

$$\varphi_\pm(\overline{z}_\pm z_\pm; E) = V_\pm(\overline{z}_\pm z_\pm) + \frac{1}{2}\phi(E) + v_\pm(\overline{z}_\pm z_\pm; E), \tag{4.14}$$

where the potentials $V_\pm$ are fixed smooth functions vanishing at $\overline{z}_\pm z_\pm = 0$, while the term $v_\pm$ vanishing at infinity and the constant ϕ are to be determined. Now in order to understand the time-dependent profile of Fermi sea first consider the situation in the absence of the brane as described in [16].

$$\langle \Psi_-^{E_-} | \Psi_+^{E_+} \rangle = \mathcal{N} e^{-i\phi} \int_0^\infty \frac{d\overline{z}_+ d\overline{z}_- dz_- dz_+}{\sqrt{\overline{z}_+ z_+}\sqrt{\overline{z}_- z_-}} e^{i(\overline{z}_+ z_- + z_+ \overline{z}_-)} e^{-i\varphi_+(z_+) - i\varphi_-(z_-)} (\overline{z}_+ z_+)^{iE_-} (\overline{z}_- z_-)^{iE_+}, \tag{4.15}$$

where $\mathcal{N}$ is the normalization. At $\beta \to \infty$ Fermi profile can be obtained from saddle point approximation which is given by

$$\overline{z}_+ z_- + z_+ \overline{z}_- = -E_\pm + (z_\pm \partial_\pm + \overline{z}_\pm \overline{\partial}_\pm)\varphi_\pm(\overline{z}_\pm z_\pm). \tag{4.16}$$

So following [16] it appears the perturbed state will be an eigenstate of the deformed hamiltonian $H = H_o + H_p$ where H_p is given by

$$H_p = (z_\pm \partial_\pm + \overline{z}_\pm \overline{\partial}_\pm)\varphi_\pm(\overline{z}_\pm z_\pm; H) \tag{4.17}$$

Now in the presence of the D brane scattering matrix element will be given by

$$\langle \overline{z}_+ z_+, E_+, \text{out} | e^{\hat{W}(t_o)} |, E_-, \overline{z}_- z_-, \text{in} \rangle$$

. S–matrix element is expressed as

$$\begin{aligned} S_{perturb} &= e^{-i\phi} \mathcal{N} \int_0^\infty \frac{d\overline{z}_+ d\overline{z}_- dz_- dz_+}{\sqrt{\overline{z}_+ z_+}\sqrt{\overline{z}_- z_-}} e^{i(\overline{z}_+ z_- + z_+ \overline{z}_-)} e^{-i\varphi(\overline{z}_- z_-)} (\overline{z}_- z_-)^{iE_-} \\ &\quad [e^{-i\int_{t_o}^\infty \hat{H}_o} [e^{-iW_{\text{proj}}(t_o)}] e^{i\int_{-\infty}^{t_o} \hat{H}_o}] e^{-i\varphi_+(\overline{z}_+ z_+)} (\overline{z}_+ z_+)^{iE_+} \\ &\sim e^{-i\phi} \mathcal{N} \int_0^\infty \frac{d\overline{z}_+ d\overline{z}_- dz_- dz_+}{\sqrt{\overline{z}_+ z_+}\sqrt{\overline{z}_- z_-}} e^{i(\overline{z}_+ z_-(t) + z_+ \overline{z}_-(t))} \\ &\quad e^{-i\varphi(\overline{z}_- z_-(t))} (\overline{z}_- z_-(t))^{iE_-} [e^{iW(t)(\overline{z}_\pm z_\pm, H_o)}] e^{-i\varphi_+(\overline{z}_+ z_+(t))} (\overline{z}_+ z_+(t))^{iE_+} \end{aligned} \tag{4.18}$$

So in the presence of the D brane, in the classical regime from (4.18) we can write the Fermi profile in the presence of D brane as

$$\overline{z}_+ z_- + z_+ \overline{z}_- = -E_\pm + (z_\pm \partial_\pm + \overline{z}_\pm \overline{\partial}_\pm)\varphi_\pm(\overline{z}_\pm z_\pm) + (z_\pm \partial_\pm + \overline{z}_\pm \overline{\partial}_\pm) W(\overline{z}_\pm z_\pm, E) \tag{4.19}$$

The perturbed hamiltonian for the deformed state is given by

$$H_p' = H_p + (z_\pm \partial_\pm + \overline{z}_\pm \overline{\partial}_\pm) W(\overline{z}_\pm z_\pm; H) \tag{4.20}$$

So essentially the Dbrane act as a source at t_o. So as we see in the presence of D brane Fermi profile develops instability.

5 Perturbation by momentum modes

5.1 Collective field theory analysis

In this subsection we will consider type 0A matrix model with the time t compactified on a circle of radius R, perturbed by momentum modes $V_{\frac{n}{R}}$ in the presence of D brane. We know that the presence of D brane change the tachyonic background [18] so that we need to go through collective field theory analysis to understand how the MQM wave function in a perturbed background with a D brane is related to the one without a brane.

Here first we briefly review the scenario without D brane and then study what happens when we consider the theory in the presence of D brane. The tachyon modes of the closed string theory are the asymptotic states of collective field theory [25]. The discrete tachyonic operator $\mathcal{T}_n \sim \int_{\text{worldsheet}} e^{\pm inx/R} e^{(|n|/R-2)\phi}$ corresponds to the following operator in matrix model [26, 28],

$$V_{\pm n/R} = e^{-\frac{n}{R}t}(\overline{z}_\pm z_\pm)^{n/R}. \tag{5.1}$$

These operators creates a discrete tachyonic state of momenta $\frac{n}{R}$ over the matrix model ground state and are periodic in Euclidean time.

$$[H_o, V_{\pm n/R}] = \mp i\frac{n}{R}V_{\pm n/R} \quad ; \quad k \geq 1. \tag{5.2}$$

So $V_{\pm n/R}$ shift the energy $E \to E \mp i\frac{n}{R}$ cause a time-dependent perturbation to Fermi sea. The perturbed state in general can be expressed as

$$\Psi^E_\pm = e^{\mp i\varphi(z_\pm \bar{z}_\pm; E)}\psi^E_{o\pm} \equiv \mathcal{W}_\pm \psi^E_{o\pm}, \tag{5.3}$$

where the phases $\varphi_\pm$ have Laurent expansion

$$\varphi_\pm(z_\pm\overline{z}_\pm; E) = \frac{1}{2}\phi(E) + R\sum_{k\geq 1} t_{\pm k}(z_\pm\overline{z}_\pm)^{k/R} - R\sum_{k\geq 1}\frac{1}{k}v_{\pm k}(z_\pm\overline{z}_\pm)^{-k/R}.. \tag{5.4}$$

$t_{\pm k}$ parametrize the asymptotic perturbation by momentum modes of NS-NS scalars, corresponding to the operator introduced (5.1), Note the above wave function asymptotically behave as

$$\Psi^E_\pm(\overline{z}_\pm z_\pm) \sim (\overline{z}_\pm z_\pm)^{\pm iE-\frac{1}{2}}\ e^{\mp\frac{1}{2}i\phi(E)}\ e^{iU_\pm(\overline{z}_\pm z_\pm)} \quad ; \quad U_\pm(\overline{z}_\pm z_\pm) = \sum_{k\geq 1}|\overline{z}_\pm z_\pm|^{\frac{k}{R}}. \tag{5.5}$$

From the above its evident that tachyonic perturbation can be achieved by deforming the integration measures $d[\overline{z}_\pm z_\pm]$ to [14]

$$[d\overline{z}_\pm z_\pm] \to [d\overline{z}_\pm z_\pm]\exp\left(\pm iU_\pm(\overline{z}_\pm z_\pm)\right). \tag{5.6}$$

Extending the discussions of section 3, these wave functions (5.3) diagonalizes the deformed kernel (5.6). While the perturbed wave function evolves in time with H_o, but

it can be seen as the stationary state w.r.t an effective hamiltonian $H = H_o + H_p(H)$, where the expressions for perturbed hamiltonian H_p from semiclassical analysis is obtained in [16] as we reviewed in section 4. The partition function is given by $\mathrm{Tr} e^{-2\pi R\beta H}$, following section 3 which can also be expressed as Fredholm determinant. We have the free energy$\mathcal{F} = -i\sum_{r\geq 1/2}\phi\left(ir/R-\mu\right)$ where $\phi(E)$ is the phase described by (5.4). It satisfies the equation

$$\phi(-\mu) = 2\sin\left(\frac{\partial_\mu}{2\beta R}\right)\mathcal{F}(\mu, R).. \tag{5.7}$$

Having given a brief review of type 0A MQM perturbed by the momentum modes lets consider the theory with D brane. Note that type 0A MQM without any net D0 brane background charge, in the double scaling limit can be viewed as a pair of noninteracting fermions moving in inverted harmonic oscillator potential in x and y direction respectively (2.8). So from the analysis of [25] its apparent that the respective collective field theory will be the generalization of the one for $c = 1$ case to two (target space) dimension(x,y) and the collective field expressed as $\phi(x, y, t) = \phi(z, \overline{z}, t)$, give the eigenvalue density in two dimension with appropriate normalization. Collective field hamiltonian will be the sum of the hamiltonian for $\phi(x, t)$ and $\phi(y, t)$. The fluctuation of the collective field over the static value $\phi = \phi_o + \partial_\tau\eta(\tau, t)$,where $\pi\phi_o = p_o = \sqrt{\mu_F - x^2 - y^2}$ with $p_o^2 = p_{ox}^2 + p_{oy}^2$; $p_{ox} = \frac{dx}{d\tau}$; $p_{oy} = \frac{dy}{d\tau}$ and $\partial_\tau\eta(\tau, t) = \psi^\dagger\psi$ where ψ corresponds to the 2nd quantized fermionic field. Now the presence of D brane implies, inclusion of a macroscopic loop operator W (Laplace transformed, localized in time) to the collective field theory action which essentially creates a coherent state over MQM ground state. In linearized approximation we have $W(z, \overline{z}, t_o) \sim \int dt \int d\tau e^{-l\overline{z}z}\partial_\tau\eta(\tau, t)\delta(t - t_o)$. So the presence of the D brane implies a field independent source term in the collective field equation of motion (which can be viewed as the back-reaction due to the D-brane). Consequently we obtain the solution for collective field at $t \geq t_o$ as $\phi = \phi_{\text{free}} + \phi_{\pm\text{perturbed}}$ where $\phi_{\text{perturbed}} = \int d\tau j(t_o, \tau)G(t_o, \tau)$ with $j(t_o)$ is the current associated with the macroscopic loop operator and G is the Green's function. As $\phi_{\text{perturbed}}$ is independent of ϕ so we see that the effect of macroscopic loop operator in the action is to change the momentum associated with η by the external current or in other words the classical solution for η will get a field independent(but profile $(x(t_o), y(t_o))$ dependent) shift. The stationary field ϕ_o will also be shifted due to the change of the potential. Essentially from string theory side we can just identify the current j (transformed from τ to ϕ space,and Fourier transformed to momentum space) with the overlap amplitude $\langle V(p)|B\rangle$ and the interaction term introduced in the collective field action $\int dtd\tau\, j\,\partial_\tau\eta\delta(t - t_o)$ as $\int \phi_{\text{cl}}(p)\langle V|B\rangle$ where ϕ_{cl} can be viewed as closed string field. On this identification we see that quadratic action for η [25, 28] along with the source term resembles the closed string field action in the presence of D brane $S = S_{\text{closed}}(\phi_{\text{cl}}) + \phi_{\text{cl}}(X, \phi)\langle V(X, \phi)|B\rangle$ where ϕ_{cl} is the closed string field. Change in momentum of η due to interaction with the localized source has an explanation in the

fact that closed string momentum is not conserved in the direction of spacelike boundary condition of D brane. Now the meaning of the constraint (2.25) is that we have to ensure the fact that while the collective field is in interaction with the localized source $L_{\rm int} = \int d\tau dt\, j\,(\partial_\tau \eta)(t.\tau)\,\delta(t-t_o)$, the hamiltonian for η will always remain conserved which implies time translation invariance of complete action with no leakage from the bulk by making $\delta_t L_{\rm int}|_{\rm t=t_o} = 0$. So the quantized action for η always gives the propagator have a pole corresponding to 2D massless scalar [25] i.e resembles the tachyon. So from string theoretical point of view we see that the constraint act as a no leakage condition ensures bulk conformal invariance as we mentioned in section 2.

. Now lets find out the exact form of MQM wave function in the presence of D brane in the background perturbed by momentum modes from Collective field theory(which in the absence of Dbrane is given by (5.3)). Lets proceed in the following way.

The collective field equation of motion implies the classical solution for left and right moving field $\alpha_{x\pm}$, $\alpha_{y\pm}$ (given by $\phi(x) + \alpha'\partial_x \Pi(x)$, $\phi(y) + \alpha'\partial_y \Pi(y)$ where Π is the collective field momentum), correspond to fermionic momentum density at the edge of the Fermi sea [26, 28]. From the expression of the hamiltonian (2.8), Fermi surface is described by

$$\frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{4\alpha'}(\hat{x}^2 + \hat{y}^2) = -\mu. \tag{5.8}$$

The collective field equation corresponds to two separate equation for two fermions described by x,y as no interaction exists among them. So the momentum density $p_{\pm x}, p_{\pm y}$ at the edge of the Fermi surface evolves with time t as [26]

$$\partial_t p_{\pm x} + p_{\pm x}\partial_x p_{\pm x} - x = 0 \quad ; \quad \partial_t p_{\pm y} + p_{\pm y}\partial_y p_{\pm y} - y = 0. \tag{5.9}$$

If we consider the fluctuation of collective field around its classical solution $p_o\,(p_{ox}, p_{oy})$. $\alpha_\pm \to p_o + \epsilon_\pm$ defining $\epsilon_\pm = \frac{1}{p_o}\xi_\pm$, $\xi_\pm$ is shown to correspond the right and left moving tachyonic fluctuations [28]. Now at classical limit the macroscopic loop operator contributes a source term to the above

$$\partial_t p_{\pm x} + p_{\pm x}\partial_x p_{\pm x} - x = \left[\frac{\partial W(x,y)}{\partial x}|_y\right]\delta(t-t_o) \quad ; \quad \partial_t p_{\pm y} + p_{\pm y}\partial_y p_{\pm y} - y = \left[\frac{\partial W(x,y)}{\partial y}|_x\right]\delta(t-t_o). \tag{5.10}$$

This essentially gives a discontinuity

$$p_{\pm x_i}|_{t_o+\epsilon} - p_{\pm x_i}|_{t_o-\epsilon} = \frac{\partial W(x,y)}{\partial x_i}|_{t_o} \quad , \tag{5.11}$$

where in (5.11) x_i stands for x,y for $i = 1, 2$. So the effect of D brane is to change the Fermi profile above $t \geq t_o$ to $p_o \to p'_o = p_o + \frac{\partial W(x,y)}{\partial x_i}(t_o)$. As above t_o, $p_\pm$ evolves according to free hamiltonian H_o so we can express the fluctuation of the collective field for $t \geq t_o$ as $\epsilon'_\pm = \frac{1}{p'_o}\xi'_\pm$ to see $\xi'_\pm$ corresponds to tachyonic fluctuation mode. However the redefinition above t_o implies a nonlinear shift of collective field momenta $p'_o = p'_o(p_o, x, y)$ and the fluctuation $\xi' = \xi'(\xi, p_o, x, y)$. $\epsilon'_{\pm x}$,$\epsilon'_{\pm y}$ combined to give complex tachyonic field.

So from the viewpoint of collective field scenario lets find out the picture in MQM. The shift of fluctuation mode above $\xi \to \xi'$ implies a shift in the perturbing phase (5.4) .

$$\psi'^{\;E}_{p\pm>} = e^{\mp i\varphi_w(z_\pm \overline{z}_\pm;E)}\psi^E_{o\pm} = \mathcal{W}'\psi^E_{o\pm}, \tag{5.12}$$

where

$$\begin{aligned}\varphi_{w\pm}(z_\pm \overline{z}_\pm;E) = \frac{1}{2}\phi(E) + R\sum_{k\geq 1} t_{\pm k}(t_{m\pm}, v_{n\pm}, \frac{r}{R}, E) f_{tk}(E, \overline{z}_\pm z_\pm)\left[z_\pm \overline{z}_\pm\right]^{k/R} \\ - R\sum_{k\geq 1}\frac{1}{k} v_{\pm k}(t_{m\pm}, v_{n\pm}, \frac{r}{R}, E) f_{vk}(E, \overline{z}_\pm z_\pm)\left[z_\pm \overline{z}_\pm\right]^{-k/R},\end{aligned} \tag{5.13}$$

f_{vk} and $f_{tk}$is the extra factor arises due to the nonlinear shift in ξ from $\overline{z}_\pm z_\pm$ factor which arises due to the action of W. These factors give nonperiodic shift to momentum by integer numbers i.e $\frac{k}{R} \to \frac{k}{R} + n$ with $t_{\pm k} = t_{\pm k}(t_{m\pm}, v_{n\pm}, E)$, $v_{\pm k} = v_{\pm k}(t_{m\pm}, v_{n\pm}, E)$. Note that once we try to interpret the consequence of (5.11) in MQM we need to replace $W \to W_{\text{proj}}$ (2.41). The fact that in absence of $\hat{W}$ we get back our original wave function so (5.3) implies that the shifted dressing operator $\mathcal{W}'$ is of the form $\mathcal{W}' = F(\hat{W})\mathcal{W}$ with $F \to 1$ in absence of $\hat{W}$. Exploiting the fact that (2.39) is just the quantum version of (5.11) and the indication of the semiclassical analysis (4.19) implies $\mathcal{W}'$ can be expressed as :

$$\Psi'^{E}_{p\pm>} = e^{\mp i\varphi_w(\overline{z}_\pm z_\pm;E)}\psi^E_o = (1 \mp \hat{W}_{\text{proj}})e^{\mp i\varphi(\overline{z}_\pm z_\pm;E)}\psi^E_{o\pm} = (1 \mp \hat{W}_{\text{proj}})\mathcal{W}\psi^E_{o\pm}, \tag{5.14}$$

This is because at double scaling limit $\beta \to \infty$ the wave function is effectively given by $e^{-\hat{W}}\mathcal{W}\psi^E_{o\pm}$(as we have seen in (3.34)). This is the solution from (2.40) if the initial free fermionic wave function (2.28) is replaced by the dressed one (5.3). Finally as before we will express W_{proj} in terms of $\hat{\overline{z}}_-\hat{z}_-$ ($\hat{\overline{z}}_+\hat{z}_+$) in $\overline{z}_+z_+$ ($\overline{z}_-z_-$) basis as this will express (5.13) in the following form

$$\begin{aligned}\varphi_{wp\pm}(z_\pm \overline{z}_\pm;E) = \frac{1}{2}\phi(E) + R\sum_{k\geq 1} t_{\pm k}(t_{m\pm}, v_{n\pm}, \frac{r}{R}, E) f_{tk}(E, a_r(\overline{z}_\pm z_\pm)^{-r})\left[z_\pm \overline{z}_\pm\right]^{k/R} \\ - R\sum_{k\geq 1}\frac{1}{k} v_{\pm k}(t_{m\pm}, v_{n\pm}, \frac{r}{R}, E) f_{vk}(E, b_r(\overline{z}_\pm z_\pm)^{-r})\left[z_\pm \overline{z}_\pm\right]^{-k/R},\text{‘}\end{aligned} \tag{5.15}$$

where r is a positive integer. In next subsection we will see that this expression of φ' lead to convergence of the partition function. The tachyonic perturbation can be introduced in the path integral by deforming the kernel as in (5.6) and consequently the string partition function can be expressed as the Fredholm determinant as in (3.27). We can evaluate the Fredholm determinant with a set of diagonalizing wave function which is given by (5.12). In the next part of this section we will evaluate the gran canonical partition function in the hamiltonian formalism. We will show that the tachyonic deformation in the presence of the D brane is generated by a system of commuting flows H_n associated with the coupling constants $t_{\pm n}$. The associated integrable structure of the partition function is that of a constrained Toda Lattice hierarchy. Now in order to see the Toda structure of the partition function we need to review Lax formalism.

5.2 Lax Formalism

Here we will briefly review the Lax formalism in the context of Type 0A matrix model. Consider the operator $(\hat{z}_\pm\hat{\overline{z}}_\pm)$ which can be represented as shift operators $\hat{\omega}^{\pm 1}$, where $\hat{\omega}$ acts on energy eigenstates as $\hat{\omega}^{\pm 1}\psi^E_\pm = \psi^{E\mp i}_\pm$. We have $\hat{\omega} = e^{-i\partial_E}$ shifts the variable E by i. The operators $\hat{\omega}$ and $\hat{E}$ satisfy the Heisenberg-Weyl commutation relation

$$[\hat{\omega}, -\hat{E}] = i\hat{\omega}, \qquad [\hat{\omega}^{-1}, -\hat{E}] = -i\hat{\omega}^{-1}. \tag{5.16}$$

Now let us consider the representation of these commutation relations in the perturbed theory. The dressing operators $\mathcal{W}_\pm$ (5.5)are now exponents of series in $\hat{\omega}$ with $\hat{E}$-dependent coefficient

$$\hat{\mathcal{W}}_\pm = e^{iR\sum_{n\geq 1} t_{\pm n}\hat{\omega}^{n/R}} \; e^{\mp i\phi(E)} e^{iR\sum_{n\geq 1} v_{\pm n}(E)\ \hat{\omega}^{-n/R}}. \tag{5.17}$$

The operators

$$\begin{aligned} L_+ &= \mathcal{W}_+\hat{\omega}\mathcal{W}_+^{-1}, \quad L_- = \mathcal{W}_-\hat{\omega}^{-1}, \mathcal{W}_-^{-1}, \\ M_+ &= -\mathcal{W}_+\hat{E}\mathcal{W}_+^{-1} \quad M_- = -\mathcal{W}_-\hat{E}, \mathcal{W}_-^{-1}. \end{aligned} \tag{5.18}$$

known as Lax and Orlov-Schulman operators satisfy the same commutation relations as the operators $\hat{\omega}$ and $\hat{E}$

$$[L_+, M_+] = iL_+ \quad , \quad [L_-, M_-] = -iL_-. \tag{5.19}$$

The Lax operators $L_\pm$ represent the canonical coordinates $\hat{\overline{z}}_\pm\hat{z}_\pm$ in the basis of perturbed wave functions

$$\langle E|e^{\pm i\frac{\phi_0}{2}}\hat{\mathcal{W}}_\pm L_\pm|\overline{z}_\pm z_\pm\rangle = \langle E|e^{\pm i\frac{\phi_0}{2}}\hat{\mathcal{W}}_\pm\hat{\overline{z}}_\pm\hat{z}_\pm|\overline{z}_\pm z_\pm\rangle, \tag{5.20}$$

while the Orlov-Shulman operators $M_\pm$ represent hamiltonian $H_0 = -\frac{1}{2}(\hat{\overline{z}}_+\hat{z}_- + \hat{\overline{z}}_-\hat{z}_+)$. Therefore the L and M operators are related also by

$$M_+ = M_-, \quad [L_+, L_-] = 2iM_\pm, \quad \{L_+, L_-\} = 2M_\pm^2 - \frac{1}{2}. \tag{5.21}$$

The last identity is not satisfied automatically in the Toda system and represent an additional constraint analogous to the string equations. The operators $M_\pm$ can be expanded as infinite series of the L-operators. Indeed, as they act to the dressed wave functions as

$$\begin{aligned} \langle E| \quad & e^{\pm i\phi_0}\hat{\mathcal{W}}_\pm\ M_\pm|\overline{z}_\pm z_\pm\rangle = \pm i(z_\pm\partial_{z_\pm} + \overline{z}_\pm\partial_{\overline{z}_\pm} + 1)\Psi^E_\pm(\overline{z}_\pm z_\pm) \\ &= \left(\sum_{k\geq 1} kt_{\pm k}(\overline{z}_\pm z_\pm)^{k/R} + \mu + \sum_{k\geq 1} v_{\pm k}\overline{z}_\pm z_\pm{}^{-k/R}\right)\Psi^E_\pm((\overline{z}_\pm z_\pm)). \end{aligned} \tag{5.22}$$

we can write

$$M_\pm = \sum_{k\geq 1} kt_{\pm k}L_\pm^{k/R} + \mu + \sum_{k\geq 1} v_{\pm k}L_\pm^{-k/R}. \tag{5.23}$$

In order to exploit the Lax equations and the string equations we need the explicit form of the two operators. It follows from that $L_{\pm}$ can be represented as series of the form

$$L_{+} = e^{-i\phi/2}\left(\omega + \sum_{k\geq 1} a_k \omega^{1-n/R}\right) e^{i\phi/2},$$
$$L_{-} = e^{i\phi/2}\left(\omega^{-1} + \sum_{k\geq 1} a_{-k} \omega^{-1+n/R}\right) e^{-i\phi/2}. \tag{5.24}$$

Recall that the dressing operators $\mathcal{W}_{\pm}$ in terms of $\hat{E}$ and $\hat{\omega}$ are of the for

$$\mathcal{W}_{\pm} = e^{\mp i\phi/2}\left(1 + \sum_{k\geq 1} w_{\pm k}\hat{\omega}^{\mp k/R}\right) e^{\mp iR\sum_{k\geq 1} t_{\pm k}\hat{\omega}^{\pm k/R}} \tag{5.25}$$

Studying the evolution laws of the Orlov–Shulman operators, one can find that [20]

$$\frac{\partial v_k}{\partial t_l} = \frac{\partial v_l}{\partial t_k}. \tag{5.26}$$

It means that there exists a generating function $\tau_s[t]$ of all coefficients $v_{\pm k}$

$$v_k(s) = (\frac{1}{\beta})^2 \frac{\partial \log \tau_s[t]}{\partial t_k}. \tag{5.27}$$

It is called τ-*function* of Toda hierarchy. It also allows to reproduce the zero mode ϕ and, consequently, the first coefficient in the expansion of the Lax operators

$$e^{\beta\phi(s)} = \frac{\tau_s}{\tau_{s+\frac{1}{\beta}}}, \qquad r^2(s-\frac{1}{\beta}) = \frac{\tau_{s+\frac{1}{\beta}}\tau_{s-\frac{1}{\beta}}}{\tau_s^2}. \tag{5.28}$$

We are going to show that the partition function coincides with τ–function (5.27). Finally note as the partition function is described in terms of the Fermi level μ. So in the description of Lax formalism we will replace E by μ. Now let us discuss about the integrable flow. Let us identify the integrable flows associated with the coupling constants t_n.

$$\partial_{t_n} L_{\pm} = [H_n, L_{\pm}], \tag{5.29}$$

where from (5.18), the operators H_n are related to the dressing operators as

$$H_n = (\partial_{t_n}\mathcal{W}_{+})\mathcal{W}_{+}^{-1} = (\partial_{t_n}\mathcal{W}_{-})\mathcal{W}_{-}^{-1}. \tag{5.30}$$

it is clear that $H_n = W_{+}\hat{\omega}^{n/R}W_{+}^{-1}+$ negative powers of $\hat{\omega}^{1/R}$, which implies expression of H_n can be given by [14]

$$H_{\pm n} = (L_{\pm}^{n/R})_{\gtrless} + \frac{1}{2}(L_{\pm}^{n/R})_0, \qquad n>0, \tag{5.31}$$

$$\partial_{t_m} H_n - \partial_{t_n} H_m - [H_m, H_n] = 0. \tag{5.32}$$

Equations (5.32,5.30,5.31) imply that the perturbed theory possesses the Toda lattice integrable structure. The Toda structure implies an infinite hierarchy of PDE's for the coefficients v_n of the dressing operators, the first of which is the Toda equation for the phase $\phi(\mu) \equiv \phi(E=-\mu)$

$$i\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_{-1}}\phi(\mu) = e^{i\phi(\mu)-i\phi(\mu-i/R)} - e^{i\phi(\mu+i/R)-i\phi(\mu)}. \tag{5.33}$$

5.3 String theory on a circle with the D brane in the presence of tachyonic background

In this section we are going to evaluate the free energy of type 0A MQM in the presence of D brane and with tachyonic background, in the grand canonical ensemble and try to understand the relevant string theory. Recall in section 3 we have seen that in the absence of the momentum modes, within the time circle $0 \le t \le 2\pi R$ the solution of the Schrodinger equation corresponds to the free fermionic wave function. This in the string theory side giving a picture that we have free closed string states along the circle and the coherent states are strongly localized at $t = t_o \equiv iX^o$. So with the same view in the presence of tachyonic background, within the circle $0 \le t \le 2\pi R$ the wave function must be given by (5.3). The perturbed wave function, while time dependent w.r.t the free hamiltonian H_o it is stationary w.r.t an effective hamiltonian H, similar as discussed in section 5.1. So lets consider the perturbed MQM with the effective hamiltonian $H = H_o + H_p(H)$ in the presence of D brane. First consider the partition function (3.7) where now we replace the integration kernels with the deformed measures (5.6). So as a generalization of (3.7), in the perturbed background, the Matrix model partition function in the presence of D brane (3.23, 3.24,3.25) will be with the deformed kernel as

$$\begin{aligned}\mathcal{Z}_N(t) = &\int_{-\infty}^{\infty}\prod_{k=1}^{N}[dz_{+k}][dz_{-k}][d\overline{z}_{+k}][d\overline{z}_{-k}][dt_k]e^{it_{n\pm}(\overline{z}_\pm z_\pm)^{\frac{n}{R}}}\det_{jk}\left(e^{it_{jk}\overline{z}_{+j}z_{-k}}\right)\\ &\det_{jk}\left(e^{-iqz_{+j}\overline{z}_{-k}}\right)\det_{jk}\left(e^{it_{jk}^{-1}\overline{z}_{+j}z_{-k}}\right)\ \det_{jk}\left(e^{-iqz_{+j}\overline{z}_{-k}}\right)\\ &\exp[\sum_i \log(1+\frac{\overline{z}_{+i}z_{+i}+\overline{z}_{-i}z_{-i}+\overline{z}_{+i}z_{-i}+\overline{z}_{-i}z_{+i}}{\mu_B^2})].\end{aligned} \tag{5.34}$$

The partition function will be given by Fredholm determinant $\mathrm{Det}(1+e^{-\beta\mu}WK))$ (3.27) where in order to evaluate the determinant we need to choose the basis which diagonalizes K (i.e (3.13) with deformed measure as given in (5.34)) and we evaluate the expectation value of $\hat{W}$ in the same. In order to evaluate free energy in the presence of D brane we will proceed in the following way. First consider the scenario without D brane. Recall the expression of free energy which is expressed in terms of the phase of wave function [14, 15](which is of the same form of (3.42), expressed in the absence of brane). In a

perturbed background the phase $\phi(E)$ will be replaced by that of the perturbed wave function (5.3) in the expression of free energy [14]. So for the effective hamiltonian H ($H = H_o + H_p(H)$ where $H_p$in the semiclassical limit obtained in (4.17)) of which (5.3) is an eigenfunction, the analysis of section 3.3 implies that free energy of the perturbed system in grand canonical ensemble is given by $\mathcal{F} = \log\mathcal{Z}$ with $\mathcal{Z} = \sum_{N=0}^{\infty} e^{-2\pi R\beta\mu N}\{\mathrm{Tr}e^{-2\pi R\beta H}\}_N =$ Det $(1 + e^{-2\pi R\beta(\mu+H)})$. This is supported from the view of [16] where in the semiclassical regime the explicit expression of $\mathcal{Z}$ is obtained in this form. The Fredholm determinant (3.13) in a perturbed background is given by $\mathcal{Z}$ [14]. In the presence of D brane we have the Fredholm determinant (3.27) which in perturbed background is expressed in (5.34). So as in section 3.3 free energy must be obtained from the thermal partition function in the presence of D brane i.e by insertion of the operator $e^{W(t_o)}$ in the partition function and evaluating the expectation value. So here in hamiltonian formalism we will evaluate the grand canonical partition function $\mathrm{Det}(1 + e^{-\beta(H+\mu)})$ in the basis (5.3) with the insertion of the operator and it must be same as the Fredholm determinant (5.34). Here we are going to show that if we consider the prtojected theory as described in section 2, the above mentioned grand canonical partition function have the integrable structure of tau function of Toda hierarchy. Now the partition function with the momentum modes in the presence of the D brane is given by the transition amplitude from the initial state $\mathcal{W}\psi_o$ to the final state$\mathcal{W}'\psi_{o>}$, where $\psi_{o>}$ is given in (2.41) and they represent the fermionic wave function before and after being scattered from the D brane and the corresponding dressing operator is $\mathcal{W}'$ .Now note that in a compact dimension just before being scattered, the wave function at $t = 2\pi R - \epsilon$ must be given by the one at $t = \epsilon$ with a time evolution $2\pi R$. This leads to the identity

$$\mathcal{W}'\psi_{o>}(t_o) = \mathcal{W}'(1 - W(\hat{\bar{z}}_\pm \hat{z}_\pm, H_o))\psi_o(t_o) = \Big(1 - W(\hat{\bar{z}}_\pm \hat{z}_\pm, H_o)\Big)\,\mathcal{W}\psi_o(t_o), \tag{5.35}$$

where $t_o = 0 \equiv 2\pi R$. The above relation can also be viewed from (2.40) in the presence of tachyonic background, if we replace the initial fermionic wave function (2.28) by the dressed one (5.3). Hence the partition function on the circle corresponds to the transition amplitude

$$\begin{aligned}
\mathcal{Z} &= \lim_{\epsilon\to 0} \langle \mathcal{W}\psi_o(\epsilon)|\mathcal{W}'\psi_{o>}(2\pi R - \epsilon)\rangle \\
&= \lim_{\epsilon\to 0} \langle \mathcal{W}\psi_o(\epsilon)|(1 - W(\hat{\bar{z}}_\pm \hat{z}_\pm, H_o))|\mathcal{W}\psi_o(2\pi R - \epsilon)\rangle \\
&= \mathrm{Tr}_{\mathcal{W}\psi_o}\{e^{-\beta[\int_\epsilon^{2\pi R-\epsilon} dtH + \int_{-\epsilon}^{\epsilon} dtH] + \int_{-\epsilon}^{\epsilon} dtW\delta(t)}\} \\
&= \mathrm{Tr}_{\mathcal{W}\psi_o}\{e^{-\beta[\int_\epsilon^{2\pi R-\epsilon} dtH]} e^{W(t=0)}\} \\
&= \mathrm{Tr}_{\mathcal{W}\psi_o}\{e^{-2\pi R\beta H} e^{W(t=0)}\}.
\end{aligned} \tag{5.36}$$

Where the partition function is evaluated in Euclidean time and $\mathrm{Tr}_{\mathcal{W}\psi_o}$ denotes the trace taken w.r.t (5.3)[17]. Grand canonical Partition function will be given by the following expression where we will have the contribution from singlet states only

$$\prod_E \{1 + e^{-2\beta\pi R(\mu+E)} \langle\psi_p^E| e^{\hat{W}(t=0)} |\psi_p^E\rangle\} = \prod_E \{1 + e^{-2\beta\pi R(\mu+E)} \langle\psi_p^E| e^{\hat{W}} |\psi_p^E\rangle\}, \tag{5.37}$$

Note, we could write the above expression for grand canonical partition function only because $\hat{W}$ can be expressed as the direct product of the operators for the single fermionic states. Now following (2.15, 2.41) the partition function (5.36) can be expressed as

$$\begin{aligned} \prod_E \quad & [1 + e^{-2\beta\pi R(\mu+E)} \langle\psi_p^E| e^{\log(1+\frac{2\hat{\overline{z}}_\pm \hat{z}_\pm - 2H_o}{\mu_B^2})} |\psi_p^E\rangle] \\ &= \prod_E [1 + e^{-2\beta\pi R(\mu+E)} \langle\psi_p^E| (1 + \frac{\hat{\overline{z}}_\pm \hat{z}_\pm - 2H_o}{\mu_B^2}) |\psi_p^E\rangle]. \end{aligned} \tag{5.38}$$

Now we have shown in the appendix that $\langle \overline{z}_+ z_+ | \hat{\overline{z}}_+ \hat{z}_+ | \overline{z}_+ z_+ \rangle$ and $\langle \overline{z}_- z_- | \hat{\overline{z}}_- \hat{z}_- | \overline{z}_- z_- \rangle$ diverge. So we must express $\hat{W}$ as $W(\hat{\overline{z}}_- \hat{z}_-, H_o)$ for the basis $|\overline{z}_+ z_+, E\rangle$ basis and vice versa. Now note according to the commutation relation (2.30) and from the form of the wave function (5.3)

$$\hat{\overline{z}}_- \hat{z}_- = e^{-i\frac{\varphi(\hat{E+i})}{2}} \frac{\partial}{\partial \overline{z}_+} \frac{\partial}{\partial z_+} e^{i\frac{\varphi(\hat{E})}{2}}$$

. So according to the analysis Appendix, $\langle\psi_p^E| \frac{2\hat{\overline{z}}_- \hat{z}_-}{\mu_B^2} |\psi_p^E\rangle = 0$ except when R is an integer[18]. However for R an integer it contributes a constant term independent of ϕ, E, in the partition function and can be ignored by subtracting out an overall constant from the hamiltonian which amounts to multiplying the partition function by an overall factor. For the macroscopic loop operator in any other sector, we can proceed in the same way . So we can write the partition function as

$$\begin{aligned} \prod_E & [1 + \langle\psi_p^E| (1 + \frac{\hat{\overline{z}}_+ \hat{z}_- + \hat{\overline{z}}_- \hat{z}_+}{\mu_B^2}) e^{-2\beta\pi R(\mu+E)}) |\psi_p^E\rangle] \\ &= \prod_E [1 + \langle\psi_p^E| e^{\log(1-2\frac{H_0}{\mu_B^2})} e^{-2\beta\pi R(\mu+E)}) |\psi_p^E\rangle] \\ &= \prod_E [1 + \langle\psi_p^E| e^{-2\beta\pi R(\mu+H-\frac{1}{2\pi\beta R}\log(1-2\frac{H_o}{\mu_B^2})} |\psi_p^E\rangle] \\ &= \prod_E [1 + \langle\psi_p^E| e^{-2\beta\pi R(\mu+H_o'+H_p)} |\psi_p^E\rangle] \\ &= \mathrm{Tr}_{\psi_p^E} [e^{-2\beta\pi R(\mu+H_o'+H_p)}] \end{aligned} \tag{5.39}$$

[17] In order to reach from the 2nd to 3rd step in (5.36)we utilize the fact that we can scale the time $t \to \beta t$ so that the term with the macroscopic loop operator $\int dt W(t)\delta(t-t_o)$ will get a factor $\frac{1}{\beta}$ so that in the double scaling limit where $\beta \to \infty$ and with Euclidean time, we can lift up the term to the exponential and the exponent gives an exact expression what we have obtained from the path integral (2.36)

[18] This is because we can write the integral as $\langle\psi_o|(\overline{z}_+ z_+)^{\frac{n}{R}-1}|\psi_o\rangle$ which following the analysis of Appendix-A contributes only at pole.

where H_o' is as discussed in (3.45), given by $H_o' = H_o - \frac{1}{2\pi R\beta}\log(1-\frac{2H_o}{\mu_B^2})$; $H_p = H_p(\overline{z}_\pm z_\pm, H)$ is the effective perturbation in the presence of momentum modes [19].

First note that $|\psi_p^E\rangle$, the eigenstate of $H = H_o + H_p$ does not diagonalize the complete effective hamiltonian $H_{\text{eff}} = H - \frac{1}{2\pi\beta R}\log(1 - 2\frac{H_o}{\mu_B^2})$. This is the indication from collective field theory as we discussed in section 5.1 that the effect of putting a macroscopic loop operator in MQM action is to deform the static Fermi sea as well as the tachyonic background which shows up as a nonlinear shift of the perturbing phase $\varphi \to \varphi_{wp}$ (5.15,5.14). Clearly the eigenfunction which diagonalizes the complete effective hamiltonian $H_o' + H_p$ will be given by a shift $\mathcal{W} \to \mathcal{W}' = e^{i\frac{\phi'}{2}}$ where $i\phi'$ will be of similar form of (5.15) and so we will denote it by ϕ_{wp}. Now we can express $\mathcal{W}'$ as the product $\mathcal{W}' = U(\hat{\overline{z}}_\pm \hat{z}_\pm, H_o)\mathcal{W}$ where U is the factor responsible for the presence of macroscopic loop operator so that in absence of the operator we should have $\mathcal{W}' = \mathcal{W}$. Now in order to diagonalize lets recall the tricks we used in (3.44). If $\mathcal{W}'\psi_o$ diagonalize the deformed hamiltonian in (5.39) then we can replace the partition function with deformed hamiltonian $H = {H_o}' + H_p'(H, \overline{z}_\pm z_\pm)$ evaluated in the basis $\mathcal{W}'\psi_o(E)$ wlth the one in the shifted basis $\mathcal{W}'\psi_o(E) \to \mathcal{W}\psi_o(E')$ evaluated w.r.t the hamiltonian $H = H_o + H_p(H, \overline{z}_\pm z_\pm)$. By this shift we can identify $U(\hat{\overline{z}}_\pm \hat{z}_\pm, H_o)$ with an operator which has an effect to shift $E' \to E$ in the eigenfunction $\psi^p(E')$. So the operator U will be of the form $U(\sum_n a_n (\hat{\overline{z}}_\pm \hat{z}_\pm)^{in} ..)U_1$, where U_1 is an operator making unitary transformation to $\mathcal{W}$. Clearly $\mathcal{W}$ and $\mathcal{W}'$ are not related by unitary transformation and so $\mathcal{W}\psi_o$ and $\mathcal{W}'\psi_o$ defines the basis of completely different Hilbert space. So in the presence of the Brane we need to evaluate the partition function in the basis $\mathcal{W}'\psi_o$ which we can view as deformed tachyonic background caused by the presence of the brane. One can verify the expectation value of macroscopic loop operator $W(\hat{\overline{z}}_\pm \hat{z}_\pm, H_o)$ in this basis effectively gets contribution from its hamiltonian part i.e $W(H_o)$ as in the case for undeformed basis (5.3) and following similar steps we will get the partition function (5.39) with a shifted basis $\mathcal{W}\psi_o \to \mathcal{W}'\psi_o$. The perturbing phase ϕ' of the shifted basis in principle can have a complex part for which we need to choose appropriate normalization. However in order to evaluate the partition function we will follow (3.44). This partition function is exactly given by the one with a shift $\psi_p'^E = \mathcal{W}'\psi_o(E) \;\Rightarrow\; \psi_p^{E'} = \mathcal{W}\psi_o(E')$ and the perturbing phase $\phi_{wp}(E) \to \phi(E(E')) = \phi'(E')$ evaluated w.r.t the effective hamiltonian $H = H_o + H_p(H, \overline{z}_\pm z_\pm)$ but without insertion of the macroscopic loop operator W, where $\phi_{wp}(E)$ is as introduced in (5.15) and $\psi_p'^E$ is the basis which diagonalizes the deformed hamiltonian $H_o' + H_p$, as we discussed. So following (3.44) we can evaluate the partition function (5.39)in the shifted basis

$$\psi_p'^E \to \psi_p^{E'} = \mathcal{W}_s \psi_o^{E'} = e^{\frac{1}{2}\phi(E(E')) + R\sum_{k\geq 1} t_{\pm k}(z_\pm \bar{z}_\pm)^{k/R} + \sum_{k\geq 1}\frac{1}{k}v_{\pm k}(E(E'))(z_\pm \bar{z}_\pm)^{-k/R}}$$

[19]In order to reach from 2nd to 3rd step we used the same tricks of section 3 which implies that around a delta function in time, we can make the time interval infinitesimally small so that we can ignore the commutator terms ($[H_o', H_o]$ +....higher commutators) what can arise on exponential as a consequence of Baker Hausdorff formula

$$\psi_o^{E'} = \quad e^{\frac{1}{2}(\phi')(E')+R\sum_{k\geq 1} t_{\pm k}(z_\pm \bar{z}_\pm)^{k/R} - R\sum_{k\geq 1}\frac{1}{k}v'_{\pm k}(E')(z_\pm \bar{z}_\pm)^{-k/R}} \psi_o^{E'}, \tag{5.40}$$

where the shifted dressing operator is given by

$$\mathcal{W}_s = e^{\frac{1}{2}\phi(E(E'))+R\sum_{k\geq 1} t_{\pm k}(z_\pm \bar{z}_\pm)^{k/R} + \sum_{k\geq 1}\frac{1}{k}v_{\pm k}(E(E'))(z_\pm \bar{z}_\pm)^{-k/R}}$$

and $\phi(E)$ is the pase for perturbed wave function (5.4). So following (3.42) we have the free energy given by

$$\begin{aligned}\mathcal{F}(\mu, R) &= \phi(E(E' = \frac{ir}{\beta R} - \mu)) \\ &= -i \sum_{r=n+\frac{1}{2}\geq 0} \phi'(\frac{ir}{\beta R} - \mu),\end{aligned} \tag{5.41}$$

where we have

$$\phi'(E') = \phi(E) \tag{5.42}$$

So we see the partition function in the presence of D brane in a background perturbed by momentum modes with compactified time is obtained from the one without D brane by the shift

$$E \to E' \quad ; \quad \mathcal{W} \to \mathcal{W}_s$$

which defines a deformed Fermi surface..

5.4 Lax formalism for Type 0A MQM in the presence of D brane

Our lesson from the previous discussion is that Toda structure for Type 0A MQM perturbed by tachyonic modes ,in the presence of D brane can be obtained when we replace

$$\begin{aligned}\mathcal{W} \to \mathcal{W}_s &= e^{iR\sum_{n\geq 1} t_{\pm n}\omega^{n/R}}\ e^{\mp i\phi'(E')}\ e^{iR\sum_{n\geq 1} v'_{\pm n}(E')\ \omega^{-n/R}}. \\ \psi_o'^{E} \to \psi_o^{E'} &= \psi_o(E - \frac{1}{2\pi R}\log(1 - \frac{2E}{\mu_B^2})),\end{aligned} \tag{5.43}$$

$$\begin{aligned}L'_+ &= \mathcal{W}_{s+}\omega\mathcal{W}_{s+}^{-1}, \quad L_- = \mathcal{W}_{s-}\omega^{-1}\mathcal{W}_{s-}{}_-^{-1}, \\ M'_+ &= \mathcal{W}_{s+}\hat{E}\mathcal{W}_{s+}{}^{-1}, \quad M_- = \mathcal{W}_{s-}\hat{E}\mathcal{W}_{s-}.\end{aligned} \tag{5.44}$$

Note the operator algebra (5.19) remains same.

$$\langle E| e^{\pm i\phi'}\hat{\mathcal{W}}_{s\pm} L'_\pm |\overline{z}_\pm z_\pm\rangle = \langle E| e^{\pm i\phi'_0}\hat{\mathcal{W}}_{s\pm}\hat{\overline{z}}_\pm \hat{z}_\pm |\overline{z}_\pm z_\pm\rangle, \tag{5.45}$$

where $\phi' = \phi(E')$. Expression of M' is as described in (5.21)

$$\begin{aligned}\langle E| \quad & e^{\pm i\phi'}\hat{\mathcal{W}}_\pm\ M'_\pm |\overline{z}_\pm z_\pm\rangle = \pm i(z_\pm \partial_{z_\pm} + \overline{z}_\pm \partial_{\overline{z}_\pm} + 1)\Psi_\pm^{E'}(\overline{z}_\pm z_\pm) \\ &= \left(\sum_{k\geq 1} k\ t_{\pm k}\ (\overline{z}_\pm z_\pm)^{k/R} + E' + \sum_{k\geq 1} v'_{\pm k}\ (\overline{z}_\pm z_\pm)^{-k/R}\right)\Psi_\pm^{E'}((\overline{z}_\pm z_\pm)).\end{aligned} \tag{5.46}$$

As the partition function described in terms of Fermi level μ

$$M'_{\pm} = \sum_{k\geq 1} k t_{\pm k} {L'}_{\pm}^{k/R} + \hat{\mu} + \sum_{k\geq 1} {v'}_{\pm k} {L'_{\pm}}^{-k/R}. \tag{5.47}$$

The structure of the integrable flow remain same. The Toda flow equation will be given by $\phi'(\mu) \equiv \phi(E' = -\mu)$

$$i\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_{-1}}\phi'(\mu) = e^{i\phi'(\mu)-i\phi'(\mu-i/R)} - e^{i\phi'(\mu+i/R)-i\phi(\mu)}. \tag{5.48}$$

Now in order to see that partition function is a tau function of Toda lattice hierarchy first note that

$$\mathcal{Z}(\mu,t) = \prod_{n\geq 0} \exp\left[i\beta\phi\left(i\frac{1}{\beta}\frac{n+\frac{1}{2}}{R} - \mu\right)\right]. \tag{5.49}$$

with Fermi level $E' = -\mu$. Now on the other hand, the zero mode of the perturbing phase is actually equal to the zero mode of the dressing operators (5.43). Hence it is expressed through the τ-function as in (5.28). Since the shift in the discrete parameter n is equivalent to an imaginary shift of the chemical potential μ, so (5.49) implies

$$e^{i\beta\phi(-\mu)} = \frac{\mathcal{Z}(\mu + \frac{i}{2R\beta})}{\mathcal{Z}(\mu - \frac{i}{2R\beta})}. \tag{5.50}$$

However from (5.28) we have

$$e^{i\beta\phi(-\mu)} = \frac{\tau_o(\mu + \frac{i}{2R\beta})}{\tau_o(\mu - \frac{i}{2R\beta})} \tag{5.51}$$

So one concludes that

$$\mathcal{Z}(\mu,t) = \tau_0(\mu,t). \tag{5.52}$$

5.5 Representation in terms of a bosonic field

Here we will study the classical limit following the analysis of [14] The momentum modes can be described as the oscillator modes of a bosonic field $\varphi(\overline{z}_+z_+, \overline{z}_-z_-) = \varphi_+(\overline{z}_+z_+) + \varphi_-(\overline{z}_-z_-)$. The bosonization formula is

$$\Psi_{\pm}^{E'=-\mu-i}(\overline{z}_{\pm}z_{\pm}) = \mathcal{Z}^{-1} e^{\pm i\varphi_{\pm}(\overline{z}_{\pm}z_{\pm})} \cdot \mathcal{Z}. \tag{5.53}$$

(Note here in the presence of FZZT brane μ corresponds to the deformed Fermi surface) where $\mathcal{Z}$ is the partition function and

$$\varphi_{\pm}(\overline{z}_{\pm}z_{\pm}) = +R\sum_{k\geq 1} t_k(\overline{z}_{\pm}z_{\pm})^{k/R} + \frac{1}{R}\partial_\mu + \mu\log\overline{z}_{\pm}z_{\pm} - R\sum_{k\geq 1}\frac{1}{k}(\overline{z}_{\pm}z_{\pm})^{-k/R}\frac{\partial}{\partial t_k}. \tag{5.54}$$

Then from (5.46) the operators $M_{\pm}$ are represented by the currents $\overline{z}_{\pm}z_{\pm}\partial_{\pm}\varphi$

$$M_{\pm}^{\dagger}\Psi_{\pm}^{E}(\overline{z}_{\pm}z_{\pm})|_{E=-\mu-i} = \mathcal{Z}^{-1}\overline{z}_{\pm}z_{\pm}\partial_{\pm}\varphi \cdot \mathcal{Z}. \tag{5.55}$$

5.6 The dispersionless (quasiclassical) limit

We consider the quasiclassical limit $\beta \to \infty$. In this limit the integrable structure described above reduces to the dispersionless Toda hierarchy where the operators $\hat{\mu}$ and $\hat{\omega}$ can be considered as a pair of classical canonical variables with Poisson bracket

$$\{\omega, \mu\} = \omega \tag{5.56}$$

Similarly, all operators become c-functions of these variables. The Lax operators can be identified with the classical phase space coordinates $\overline{z}_\pm z_\pm$, which satisfy

$$\{\overline{z}_+, z_-\} = \{z_+, \overline{z}_-\} = 1 \tag{5.57}$$

The shape of the Fermi sea is determined by the classical trajectory corresponding to the Fermi level $E' = -\mu$. So we have

$$\overline{z}_+ z_- + z_+ \overline{z}_- - \frac{1}{2\pi R\beta} \log(1 - \frac{2(\overline{z}_+ z_- + z_+ \overline{z}_-)}{\mu_B^2}) - \log\epsilon = -\mu. \tag{5.58}$$

Where $\log\epsilon$ is the cut-off cancelling the singular contribution from the point $(1 - \frac{2(\overline{z}_+ z_- + z_+ \overline{z}_-)}{\mu_B^2}) = 0$. In the perturbed theory the classical trajectories are of the form

$$\overline{z}_\pm z_\pm = L'_\pm(\omega, \mu). \tag{5.59}$$

where the functions $L_\pm$ are of the form

$$L'_\pm(\omega, \mu) = e^{\frac{1}{2}\partial_\mu \phi'}\ \omega^{\pm 1} \left(1 + \sum_{k\geq 1} a'_{\pm k}(\mu)\ \omega^{\mp k/R}\right). \tag{5.60}$$

The flows H_n become Hamiltonians for the evolution along the 'times' t_n. The unitary operators $\mathcal{W}_\pm$ becomes a pair of canonical transformations between the variables ω, μ and $L_\pm, M_\pm$. Their generating functions are given by the expectation values $S_\pm = \mathcal{Z}^{-1}\ \varphi_\pm(\overline{z}_\pm z_\pm)\ \cdot \mathcal{Z}$ of the chiral components of the bosonic field ϕ

$$S_\pm = \pm R \sum_{k\geq 1} t_{\pm k}\ (\overline{z}_\pm z_\pm)^{k/R} + \mu \log(\overline{z}_\pm z_\pm) - \phi' \pm R \sum_{k\geq 1} \frac{1}{k} v'_k\ (\overline{z}_\pm z_\pm)^{-k/R}, \tag{5.61}$$

where $v_k = \partial\mathcal{F}/\partial t_k$. The differential of the function $S_\pm$ is

$$dS_\pm = M_\pm d\log(\overline{z}_\pm z_\pm) + \log\omega\ d\mu + R \sum_{n\neq 0} H_n dt_n. \tag{5.62}$$

If we consider the coordinate ω as a function of either $\overline{z}_+ z_+$ or $\overline{z}_- z_-$, then

$$\omega = e^{\partial_\mu S_+(\overline{z}_+ z_+)} = e^{\partial_\mu S_-(\overline{z}_- z_-)}., \tag{5.63}$$

The classical string equation

$$\overline{z}_+ z_- + z_+ \overline{z}_- - \frac{1}{2\pi R} \log(1 - \frac{2(\overline{z}_+ z_- + z_+ \overline{z}_-)}{\mu_B^2}) = M_+ = M_-, \tag{5.64}$$

can be written as

$$\begin{aligned}\overline{z}_+ z_- + \overline{z}_- z_+ - \frac{1}{2\pi R}\log(1 - \frac{2(\overline{z}_+ z_- + z_+ \overline{z}_-)}{\mu_B^2}) \\ = \sum_{k\geq 1} k t_k (\overline{z}_+ z_+)^{k/R} + \mu + \sum_{k\geq 1} v_k (\overline{z}_+ z_+)^{-k/R}.\end{aligned} \tag{5.65}$$

6 Conclusion

Here we have studied Type 0A matrix model in the presence of spacelike D brane which are localized in matter direction. In matrix model this is expressed by insertion of an operator $e^{W(t_o)}$ into the path integral. When we studied the respective MQM we found by application of Ward identity that the time translation invariance of the path integral in the presence of such operator gives the signal of leakage of MQM hamiltonian. However in dual string theory this phenomenon has a meaning that closed string hamiltonian is undergoing a leakage when the string is getting scattered from Dbrane! in order to obtain right string theory picture we impose a constraint (2.25) on matrix model path integral in the presence of D brane. We explained that this condition has an effect to constrain the Hilbert space generated by macroscopic loop operator while keeping type 0A MQM unaffected. We have shown that when we impose the constraint we get the matter one point function from collective field theory. We have further shown that exactly at the point of insertion of the brane (which in string theory correspond to the point where open string ends are localized) the wave function for the right and left moving component of boundary state with any momentum appears to be identical which can be seen in matrix model as a consequence of this constraint. We also found right transition amplitude from a free fermionic state to coherent state. Next we consider type 0A MQM with the time t compactified on a circle. We have shown matrix model path integral in the presence of Dbrane can be expressed as Fredholm determinant. We evaluated the thermal partition function in grand canonical ensemble. As the theory is defined on a circle so the partition function correspond to that of a deformed Fermi surface. We have further shown that in absence of any such constraint, the partition function diverges. Theory on the circle also posses a symmetry (3.33), which is parallel to string theory as we discussed in section 2.4. This symmetry clearly indicates that coherent states are strongly localized at the point of insertion of macroscopic loop operator. Finally we considered type 0A MQM in the background of momentum modes. First in section 4 we made a semiclassical analysis, studied fermionic scattering in the presence of D brane. We found the effective hamiltonian in the perturbed background from semiclassical analysis. Its known that the presence of D brane change the tachyonic background. So from collective field theory analysis we found the right expression of MQM wave function i.e exact modification of the dressing operator in the presence of Dbrane. We derived the grand canonical partition function in the perturbed background in the presence of D brane. We have shown the

partition function corresponds to tau function of Toda hierarchy. We have also analyzed the theory in dispersionless limit.

Its interesting to study T duality between type 0A and type 0B MQM in the presence of D brane. One can also study the theory in the presence of flux background and see the consequence of the constraint.

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Appendix

A Appendix

Here we will show that type 0A MQM wave functions in the presence of D brane satisfy orthogonality and biorthogonality conditions.

A.1 Orthogonality Condition

The wave function is expected to show orthonormal property.

1. For $t < t_o$ we have the wave function ψ_o given in (2.28). One can check the orthonormality property by considering the contour integral ,and it is given by [15]

$$\langle \psi_{E'} | \psi_E \rangle = \delta(E - E'). \tag{A.1}$$

For $t > t_o$, first consider the wave function $\psi_+(z_+, \overline{z}_+, t; E)$ given in (2.42). we have

$$\langle\psi(E',t)\psi(E,t)\rangle = \int dz_+ d\overline{z}_+ [\{1 - \log(1 + \frac{W(\overline{z}_- z_-, 2H_o)}{\mu_B^2})\} e^{iE'(t-t_o)} e^{i\frac{\phi_o}{2}(E')} (\overline{z}_- z_-)^{-iE'-\frac{1}{2}}]$$
$$[\{1 - \log(1 + \frac{W(\overline{z}_- z_-, 2H_o)}{\mu_B^2})\} e^{-iE(t-t_o)} e^{-i\frac{\phi_o}{2}(E)} (\overline{z}_+ z_+)^{iE-\frac{1}{2}}]. \tag{A.2}$$

Now in order to see the orthonormal property first recall the commutation relation (2.30)

Note that $\hat{z}_+$ and $\hat{z}_-$ shifts E by -i and +i respectively.So we can write

$$\hat{\overline{z}}_+ \hat{z}_+ = e^{\frac{-i\phi_o}{2}} e^{-i\partial_E} e^{\frac{i\phi_o}{2}} \tag{A.3}$$

Similarly

$$\hat{\overline{z}}_- \hat{z}_- = e^{\frac{-i\phi_o}{2}} e^{i\partial_E} e^{\frac{i\phi_o}{2}}. \tag{A.4}$$

Now to evaluate (A.2) first consider the expression

$$\langle\psi_+|[\hat{\overline{z}}_- \hat{z}_-]^m|\psi_+\rangle = \int dz_+ d\overline{z}_+ \{e^{iE't} e^{\frac{i}{2}\phi_o(E')} (z_+ \overline{z}_+)^{-iE'-\frac{1}{2}}\}$$
$$[\frac{\partial}{\partial \overline{z}_+}\frac{\partial}{\partial z_+}]^m \{e^{-iEt+mt} e^{-\frac{i}{2}\phi_o(E-mi)} (z_+ \overline{z}_+)^{iE-\frac{1}{2}}\}. \tag{A.5}$$

For $E \neq E'$ this can just be written as

$$\begin{aligned}\langle\psi_+|[\hat{\overline{z}}_- \hat{z}_-]^m|\psi_+\rangle &= e^{-\frac{i}{2}\phi_o} e^{im\partial_E} e^{\frac{i}{2}\phi_o} \int dz_+ d\overline{z}_+ \{e^{iE't} e^{\frac{i}{2}\phi_o(E')} (z_+ \overline{z}_+)^{-iE'-\frac{1}{2}}\} \\ &\quad \{e^{-iEt} e^{-\frac{i}{2}\phi_o(E)} (z_+ \overline{z}_+)^{iE-\frac{1}{2}}\} \\ &= e^{-\frac{i}{2}\phi_o} e^{im\partial_E} e^{\frac{i}{2}\phi_o} \langle\psi_+(E')|\psi_+(E)\rangle_{E\neq E'} \\ &= 0. \end{aligned} \tag{A.6}$$

For $E = E'$ (A.2) takes the form

$$\langle\psi_+|[\hat{\overline{z}}_- \hat{z}_-]^m|\psi_+\rangle = \{e^{i\phi_o(E)} e^{-i\phi_o(E-mi)}\} \int dz_+ d\overline{z}_+ [\overline{z}_+ z_+]^{-m-1} e^{-mt}. \tag{A.7}$$

From contour integral which is 0 for $m \geq 1$. Also we conclude that $\langle\psi_+|[\hat{\overline{z}}_+ \hat{z}_+]^m|\psi_+\rangle$ and $\langle\psi_-|[\hat{\overline{z}}_- \hat{z}_-]^m|\psi_-\rangle$ diverge. Here before going to show the orthogonality lets consider the situation when m is not an integer. This kind of integration we had in the section 5 in the expression $\langle\psi_o|(\overline{z}_+ z_+)^{\frac{n}{R}-1}|\psi_o\rangle$. We can consider this as the product over two branh cut integrals z_+ and $\overline{z}_+$ and the each branch cut integral can be expressed as the sum of two standard contour integral with the cut on right and left side of the real axis $0 \geq x \geq \infty$ and $-\infty \geq x \geq 0$ respectively and with a pole at $x = 0$. One can see that the integral turns out to be zero for any noninteger $\frac{m}{R}$. We have nonzero contribution only when R is an integer and the contributing term is m=R.This term corresponds to a pure pole and give a constant contribution to the integral, Now back to the question of orthogonality.

So using (A.4 ,A.7), we can be written (A.2) as

$$\begin{aligned}\langle\psi(E',t)|\psi(E,t)\rangle &= \int dz_+ d\overline{z}_+ [\{1-\log(1+\frac{f(0,2H_o)}{\mu_B^2})\}e^{iE'(t-t_o)}e^{\frac{i}{2}\phi_o(E')}(z_+\overline{z}_+)^{-iE'-\frac{1}{2}}] \\ &\quad [\{1-\log(1+\frac{f(0,2H_o)}{\mu_B^2})\}e^{-iE(t-t_o)}e^{-\frac{i}{2}\phi_o(E)}(z_+\overline{z}_+)^{iE-\frac{1}{2}}] \\ &= \int dz_+ d\overline{z}_+ [\{1-\log(1+\frac{f(0,2E)}{\mu_B^2})\}e^{iE'(t-t_o)}e^{\frac{i}{2}\phi_o(E')}(z_+\overline{z}_+)^{-iE'-\frac{1}{2}}] \\ &\quad [\{1-\log(1+\frac{f(0,2E)}{\mu_B^2})\}e^{-iE(t-t_o)}e^{-\frac{i}{2}\phi_o(E)}(z_+\overline{z}_+)^{iE-\frac{1}{2}}]. \end{aligned} \tag{A.8}$$

Now write $\phi_o = \phi_{oRe} + i\phi_{oIm}$. Shifting $\phi_{oIm} \to \phi_{oIm} - \frac{i}{2}\log[1-\log(1+\frac{f(0,2E)}{\mu_B^2})]$ we get the orthogonal property (A.1) with

$$\psi_+(E,t) = e^{iE(t-t_o)}e^{-\frac{i}{2}\phi_{0+}(E)}(z_+\overline{z}_+)^{iE-\frac{1}{2}}, \tag{A.9}$$

where

$$\phi_{o+}(E) = \phi_o(E) - i\log[1 - log(1+\frac{f(0,2E}{\mu_B^2})]. \tag{A.10}$$

Similarly for the wave function in z_- representation we find

$$\phi_{o-}(E) = \phi_o(E) + ilog[1 - log(1+\frac{f(0,2E}{\mu_B^2})] \tag{A.11}$$

So we see the consequence of the insertion of macroscopic loop operator is that the phase of the wave function develops an imaginary part which is associated with tunneling.

A.2 Biorthogonality Relation

The wave function is expected to satisfy the following biorthogonality condition which has a consequence in the evaluation of scattering amplitude [14–16].

$$\int dz_+ dz_- d\overline{z}_+ d\overline{z}_- \overline{\psi_+^E}(\overline{z}_+ z_+, t)e^{i(\overline{z}_+ z_- + \overline{z}_- z_+)}\psi_-^{E'}(\overline{z}_- z_-, t) = \delta(E-E') \tag{A.12}$$

Now for $t \leq t_o$ we have $\psi = \psi_o$ and for which biorthogonality relation is already derived in [14], [15] giving $e^{i\phi_0(E)} = \frac{\Gamma(iE+1/2)}{\Gamma(iE+1/2)}$ [15]. For $t \geq t_o$, .Biorthogonality relation takes the form

$$\begin{aligned}\int dz_+ dz_- d\overline{z}_+ d\overline{z}_- \quad & \overline{\psi}_+^E(\overline{z}_+ z_+, t)\{1-\hat{W}(\hat{z}_+, \hat{\overline{z}}_+, H_o, t)\}e^{i(\overline{z}_+ z_- + z_+\overline{z}_-)}(1-\hat{W}(\hat{z}_+, \hat{\overline{z}}_+, H_o, t))\psi_-^{E'} \\ &= \delta(E-E'), \end{aligned} \tag{A.13}$$

In order to show this note that

$$\begin{aligned}\int dz_+ dz_- d\overline{z}_+ d\overline{z}_- \quad & \overline{\psi_+^E}(z_+, \overline{z}_+, t)\{1-\hat{W}(\hat{z}_-, \hat{\overline{z}}_-, H_o, t)\}e^{i(z_+ z_- + \overline{z}_+\overline{z}_-)}(1-\hat{W}(\hat{z}_-, \hat{\overline{z}}_-, H_o, t))\psi_-^{E'} \\ &= \int dz_+ d\overline{z}_+ \overline{\psi_+^E}(z_+, \overline{z}_+, t)\{1-\hat{W}(\hat{z}_-, \hat{\overline{z}}_-, H_o, t)\}(1-\hat{W}(\frac{\partial}{\partial z_+}, \frac{\partial}{\partial \overline{z}_+}, H_o, t))\end{aligned}$$

$$
\begin{aligned}
&\int dz_- d\overline{z}_- e^{i(\overline{z}_+ z_- + \overline{z}_- z_+)}) \psi_-^{E'}(\overline{z}_- z_-, t) \\
&= \int dz_+ d\overline{z}_+ \overline{\psi_+^E}(z_+, \overline{z}_+, t) \{1 - \hat{W}(\hat{z}_-, \hat{\overline{z}}_-, H_o, t)\} (1 - \hat{W}(\hat{z}_-, \hat{\overline{z}}_-, H_o, t)) \\
&\int dz_- d\overline{z}_- e^{i(z_+ z_- + \overline{z}_+ \overline{z}_-)}) \psi_-^{E'}(z_-, \overline{z}_-, t) \\
&= R(E) \int dz_+ d\overline{z}_+ \overline{\psi_+^E}(z_-, \overline{z}_-, t) \{1 - \hat{W}(z_-, \overline{z}_- H_o, t)\} (1 - \hat{W}(z_-, \overline{z}_-, H_o, t)) \\
\psi_-^{E'}(z_+, \overline{z}_+, t) & \\
&= R(E) e^{i\phi_{o+}} \delta(E - E'). \qquad \text{(A.14)}
\end{aligned}
$$

Where in order to come from 2nd to 3rd step we used the fact that in z_+ representation we have $\hat{z}_-, \hat{\overline{z}}_- = \frac{\partial}{\partial z_+}, \frac{\partial}{\partial \overline{z}_+}$ The integral in the 4th step, we have evaluated in (A.8) leads to the last step. In order to get the biorthogonality relation(A.13) we must need to set $e^{i\phi_{o+}(E)} = R(E)$. Compared to the $t \leq t_o$ case note the shift of $\phi_o(E)$ due to the insertion of macroscopic loop operator.

$$
R(E)\psi_+^{E'}(z_+, \overline{z}_+, t) = \int_{-\infty}^{\infty} dz_- d\overline{z}_- e^{i(z_+ z_- + \overline{z}_+ \overline{z}_-)}) \psi_-^{E'}(z_-, \overline{z}_-, t) \qquad \text{(A.15)}
$$

R(E) getting absorbed to decide ϕ_o and shifting ϕ_o according to (A.11) we get the above result.

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